Adıyaman University Journal of Science

dergipark.org.tr/adyusci

ADYUSCI

9 (1) (2019)
149-164

Reverse and Jordan $(\alpha,\beta)-$ biderivation on Prime and Semi-prime Rings

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Abstract

In this study, we prove that any nonzero reverse $(\alpha,\beta)-$ biderivation on a prime ring is $(\alpha,\beta)-$ biderivation. Also, we show that any Jordan $(\alpha,\beta)-$ biderivation on non-commutative semi-prime ring R with $char(R)\neq 2$ is an $(\alpha,\beta)-$ biderivation. In addition, we investigate commutative feature of prime ring with Jordan left $(\alpha,\alpha)-$ biderivation.

Keywords: Prime ring, Semi-prime ring, Reverse $(\alpha,\beta)-$ biderivation, Jordan $(\alpha,\beta)-$ biderivation, Jordan left $(\alpha,\alpha)-$ biderivation.

Asal ve Yarıasal Halkalarda Ters ve Jordan $(\alpha,\beta)-$bitürev

Özet

Bu çalışmada, asal halka üzerinde tanımlı sıfırdan farklı bir ters $(\alpha,\beta)-$ bitürevin aynı zamanda $(\alpha,\beta)-$ bitürev olduğu ispatlanmıştır. Ayrıca, $char(R)\neq 2$ olacak biçimdeki değişmeli olmayan bir yarı-asal R halkası üzerinde tanımlı Jordan $(\alpha,\beta)-$ bitürevin aynı zamanda $(\alpha,\beta)-$ bitürev olduğu gösterilmiştir. Bunların yanında, Jordan sol $(\alpha,\alpha)-$bitürevli asal halkaların değişmeli olma özellikleri araştırılmıştır.

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Received: 30 October 2018 Accepted: 02 June 2019

Anahtar Kelimeler: Asal halka, Yarı-asal halka, Ters $(\alpha,\beta)-$ bitürev, Jordan $(\alpha,\beta)-$ bitürev, Jordan sol (α,α) $-$bitürev.

1. Introduction

Let R be a ring and $Z(R)$ be its center. Remember that R is prime if $x_1Rx_2=(0)$ implies $x_1=0$ or $x_2=0$ for any $x_1,s\in R$. Also, R is semi-prime if $x_1Rx_1=(0)$ implies $x_1=0$ for any $x_1\in R$. For $x_1,x_2\in R$, the notation $[x_1,x_2]$ denote for commutator $x_1x_2-x_2x_1$. Following identities holds for all $x_1,x_2,x_3\in R$.

- $[x_1x_2,x_3]=x_1[x_2,x_3]+[x_1,x_3]x_2$
- $[x_1,x_2x_3]=[x_1,x_2]x_3+x_2[x_1,x_3]$

Let R be a ring and S be a subring of R. A D bi additive map from $S\times S$ into R is termed a biderivation of S if $x_2\to D(x_1,x_2)$ and $x_2\to D(x_2,x_1)$ maps are denotes derivations from S into R for all $x_1\in S$. Recall that a D map from $R\times R$ into R is termed symmetric if $D(x_1,x_2)=D(x_2,x_1)$ for all $x_1,x_2\in R$. For all $x_1,x_2,x_3\in R$, a D symmetric bi additive map from $R\times R$ into R is termed a biderivation if $D(x_1x_2,x_3)=D(x_1,x_3)x_2+x_1D(x_2,x_3)$.

Several authors have studied biderivations and investigated properties of biderivations. Also, concept of symmetric biderivation is generalized different forms in time. One of these generalizations is the generalization for Jordan derivation. An d additive map from R into R is termed a Jordan derivation if $d(x_1^2)=x_1d(x_1)+d(x_1)x_1$ for all $x_1\in R$. Similarly this definition, a J bi additive map from $R\times R$ into R is termed a symmetric Jordan biderivation if $J(a^2,x_1)=aJ(a,x_1)+J(a,x_1)a$ for all $a,x_1\in R$.

In time, researchers have introduced definitions of reverse biderivation, left (similary right) biderivation and Jordan left (similarly right) biderivation.

A D bi additive map from $R\times R$ into R is termed a symmetric reverse biderivation if $D(x_1x_2,x_3)=D(x_2,x_3)x_1+x_2D(x_1,x_3)$ for all $x_1,x_2,x_3\in R$.

A D bi additive map from $R \times R$ into R is termed a symmetric left biderivation if $D(x_1x_2, x_3) = x_1D(x_2, x_3) + x_2D(x_1, x_3)$ for all $x_1, x_2, x_3 \in R$.

A J bi additive map from $R \times R$ into R is termed a symmetric Jordan left biderivation if $J(a^2, x_1) = 2aJ(a, x_1)$ for all $a, x_1 \in R$.

In [6], Herstein showed that a Jordan derivation on R prime ring with $charR \neq 2$ is also derivation. In [3], Bresar and Vukman presented brief proof his result. Then, Daif, Haetinger and Tammam El-Sayiad showed that any reverse biderivation on R prime ring is also biderivation in [7]. They studied on Jordan biderivation and showed that Jordan biderivation on R semi-prime ring with $charR \neq 2$ is also biderivation. Also, showed that R prime ring with $charR \neq 2,3$ that contains a nonzero Jordan biderivation is also commutative.

In this paper, we study on reverse $(\alpha, \beta)-$ biderivation, Jordan $(\alpha, \beta)-$ biderivation and Jordan left $(\alpha, \alpha) -$ biderivation. Let α and β are automorphism on R.

A D bi additive map from $R \times R$ into R is termed a symmetric $(\alpha, \beta) -$ biderivation if $D(x_1x_2, x_3) = \alpha(x_1)D(x_2, x_3) + D(x_1, x_3)\beta(x_2)$ for all $x_1, x_2, x_3 \in R$.

A D bi additive map from $R \times R$ into R is termed a symmetric reverse $(\alpha, \beta) -$ biderivation if $D(x_1x_2, x_3) = D(x_2, x_3)\alpha(x_1) + \beta(x_2)D(x_1, x_3)$ for all $x_1, x_2, x_3 \in R$.

A J bi additive map from $R \times R$ into R is termed a symmetric Jordan $(\alpha, \beta) -$ biderivation if $J(a^2, x_1) = \alpha(a)J(a, x_1) + J(a, x_1)\beta(a)$ for all $a, x_1 \in R$.

A bi additive map J from $R \times R$ into R is termed a symmetric Jordan left $(\alpha, \alpha) -$ biderivation if $J(a^2, x_1) = 2\alpha(a)J(a, x_1)$ for all $a, x_1 \in R$.

In [1], authors proved that any symmetric Jordan (α, β) −biderivation (or generalized Jordan (α, β) −biderivation) on R prime ring with $charR \neq 2$ is also symmetric (α, β) −biderivation (or generalized Jordan (α, β) −biderivation). Detailed information on previous studies of ring theory and different biderivations can be obtained from [1,2,4,5,7,8].

We generalize some previous studied on biderivations to reverse $(\alpha,\beta)-$ biderivation, Jordan $(\alpha,\beta)-$ biderivation and Jordan left $(\alpha,\alpha)-$ biderivation. We prove that any nonzero reverse $(\alpha,\beta)-$ biderivation on a prime ring is $(\alpha,\beta)-$ biderivation. Also, we show that any Jordan $(\alpha,\beta)-$ biderivation on non-commutative semi-prime ring R with $charR \neq 2$ is an $(\alpha,\beta)-$biderivation. In addition, we investigate commutative feature of prime ring with Jordan left $(\alpha,\alpha)-$ biderivation.

2. Preliminaries

Lemma 2.1 [1, Lemma 4.1] Let R be a prime ring such that $charR \neq 2$ and U be a non-zero square closed Lie ideal of R. Suppose that σ,τ are endomorphisms of R. If $J: R \times R \to R$ is a symmetric generalized Jordan (σ,τ) −biderivation with associated symmetric (σ,τ) −biderivation $B: R \times R \to R$ such that $J(u^2,w) = J(u,w)(u) + (u)B(u,w)$ holds for all $u,w \in U$, then for all $u,v,w,t \in U$;

i) $J(uv+vu,w) = J(u,w)\sigma(v) + J(v,w)\sigma(u) + \tau(u)B(v,w) + \tau(v)B(u,w)$

ii) $J(uvu,w) = J(u,w)\sigma(v)\tau(u) + \tau(u)B(v,w)\sigma(u) + \tau(u)\tau(v)B(u,w)$

iii) $J(uvt+tvu,w) = J(u,w)\sigma(v)\sigma(t) + J(t,w)\sigma(v)\sigma(u) +$
$\tau(u)B(v,w)\sigma(t) + \tau(t)B(v,w)\sigma(u) + \tau(u)\tau(v)B(t,w) + \tau(t)\tau(v)B(u,w)$

iv) $\{J(uv,w) - J(u,w)\sigma(v) - \tau(u)B(v,w)\}[\sigma(u),\sigma(v)] = 0.$

Lemma 2.2 [2, Lemma 4] Let R be a 2 −torsion free semiprime ring and let $a,b \in R$. If for all $x \in R$ the relation $axb + bxa = 0$ holds, then $axb = bxa = 0$ is fulfilled for all $x \in R$.

Lemma 2.3 [7, Lemma 1] Let R be a semiprime, 2 −torsion-free ring and let T be a Lie ideal of R. Suppose that $[T,T] \subset Z$; then $T \subset Z$.

Corollary 2.4 [8, Corollary 2.1]Let R be a 2 −torsion free semiprime ring, L be a Lie ideal of R such that $L \not\subseteq Z(R)$ and let $a,b \in R$. If $aLa = 0$, then $a = 0$.

3. Results on Reverse and Jordan $(\alpha,\beta)-$ biderivation

Theorem 3.1 *If R is a prime ring and $0 \neq D$ is a symmetric reverse $(\alpha,\beta)-$ biderivation, then R is commutative. Also, D is a symmetric $(\alpha,\beta)-$ biderivation.*

Proof. Let

$D(a_1a_2,x_1) = D(a_2,x_1)\alpha(a_1) + \beta(a_2)D(a_1,x_1)$ for all $a_1,a_2,x_1 \in R$. (3.1)

Replacing a_2 by a_2a_3, $a_3 \in R$ in Equation (3.1), we get

$$D(a_1(a_2a_3),x_1) = D(a_2a_3,x_1)\alpha(a_1) + \beta(a_2a_3)D(a_1,x_1)$$

$$= D(a_3,x_1)\alpha(a_2)\alpha(a_1) + \beta(a_3)D(a_2,x_1)\alpha(a_1)$$

$$+\beta(a_2a_3)D(a_1,x_1)$$

Also, replacing a_1 by a_1a_2 and a_2 by a_3, $a_3 \in R$ in Equation (3.1), we have

$$D\big((a_1a_2)a_3,x_1\big) = D(a_3,x_1)\alpha(a_1a_2) + \beta(a_3)D(a_1a_2,x_1)$$

$$= D(a_3,x_1)\alpha(a_1a_2) + \beta(a_3)D(a_2,x_1)\alpha(a_1)$$

$$+\beta(a_3)\beta(a_2)D(a_1,x_1)$$

Using equality of two relations, we obtain $D(a_3,x_1)\alpha[a_1,a_2] = \beta[a_2,a_3]D(a_1,x_1)$ for all $a_1,a_2,a_3,x_1 \in R$.

Replacing a_3 by a_2 in above relation, we get $D(a_2,x_1)\alpha[a_1,a_2] = 0$ for all $a_1,a_2,x_1 \in R$.

Replacing a_1 by ra_1, $r \in R$ in above relation, we have

$D(a_2,x_1)r\alpha[a_1,a_2] = 0$ for all $a_1,a_2,r,x_1 \in R$.

Using primeness of R and α is automorphism, we get

$D(a_2,x_1)=0$ or $[a_1,a_2]=0$ for all $a_1,a_2,x_1\in R$.

Let $C=\{a_2\in R|D(a_2,x_1)=0,\forall x_1\in R\}$ and $E=\{a_2\in R|[a_1,a_2]=0,\forall a_1\in R\}$. C and E are subgroups of additive group R whose $R=C\cup E$, but R can't be written as a union of its two proper subgroups. Then, $R=C$ or $R=E$. From $D\neq 0$ by hypothesis, we have $R=E$ and R is commutative. Using commutativity of R, we obtain

$$\begin{aligned}D(a_1a_2,x_1) &= D(a_2,x_1)\alpha(a_1)+\beta(a_2)D(a_1,x_1)\\ &= \alpha(a_1)D(a_2,x_1)+D(a_1,x_1)\beta(a_2)\end{aligned}$$

for all $a_1,a_2,x_1\in R$. This relation gives us that D is $(\alpha,\beta)-$ biderivation.

Lemma 3.2 *If R is a ring with $charR\neq 2$ and J is a symmetric Jordan $(\alpha,\beta)-$ biderivation, then* $\big(J(a_1a_2,x_1)-\alpha(a_1)J(a_2,x_1)-J(a_1,x_1)\beta(a_2)\big)\alpha(r)\alpha([a_1,a_2])+\alpha([a_1,a_2])\alpha(r)\big(J(a_1a_2,x_1)-\alpha(a_1)J(a_2,x_1)-J(a_1,x_1)\beta(a_2)\big)=0$ *for all* $a_1,a_2,r,x_1\in R$.

Proof. Taking the element $a_1a_2ra_2a_1+a_2a_1ra_1a_2\in R$ for $a_1,a_2,r,\in R$ and using $J(a_1a_2a_1,x_1)=\alpha(a_1a_2)J(a_1,x_1)+\alpha(a_1)J(a_2,x_1)\beta(a_1)+J(a_1,x_1)\beta(a_2a_1)$ from Lemma (2.1), we have

$$J(a_1a_2ra_2a_1+a_2a_1ra_1a_2,x_1)=J(a_1(a_2ra_2)a_1+a_2(a_1ra_1)a_2,x_1)$$

$$=\alpha(a_1a_2ra_2)J(a_1,x_1)+\alpha(a_1)J(a_2ra_2,x_1)\beta(a_1)+J(a_1,x_1)\beta(a_2ra_2a_1)$$

$$+\alpha(a_2a_1ra_1)J(a_2,x_1)+\alpha(a_2)J(a_1ra_1,x_1)\beta(a_2)+J(a_2,x_1)\beta(a_1ra_1a_2)$$

for all $a_1,a_2,x_1\in R$. On the other hand, using $J(a_1a_2a_3+a_3a_2a_1,x_1)=\alpha(a_1a_2)J(a_3,x_1)+\alpha(a_3a_2)J(a_1,x_1)+\alpha(a_1)J(a_2,x_1)\beta(a_3)+\alpha(a_3)J(a_2,x_1)\beta(a_1)+J(a_1,x_1)\beta(a_2a_3)+J(a_3,x_1)\beta(a_2a_1)$ from Lemma (2.1), we have

$$J(a_1a_2ra_2a_1+a_2a_1ra_1a_2,x_1)=J\big((a_1a_2)r(a_2a_1)+(a_2a_1)r(a_1a_2),x_1\big)$$

$$=\alpha(a_1a_2r)J(a_2a_1,x_1)+\alpha(a_2a_1r)J(a_1a_2,x_1)+\alpha(a_1a_2)J(r,x_1)\beta(a_2a_1)$$

$+\alpha(a_2a_1)J(r,x_1)\beta(a_1a_2)+J(a_1a_2,x_1)\beta(ra_2a_1)+J(a_2a_1,x_1)\beta(ra_1a_2)$

for all $a_1,a_2,x_1\in R$.Using equality of two relations, we obtain

$$
\begin{aligned}
0 &= \alpha(a_2a_1ra_1)J(a_2,x_1)+\alpha(a_2a_1r)J(a_1,x_1)\beta(a_2)+\alpha(a_2)J(a_1,x_1)\beta(ra_1a_2)\\
&\quad +J(a_2,x_1)\beta(a_1ra_1a_2)+\alpha(a_1a_2ra_2)J(a_1,x_1)+\alpha(a_1a_2r)J(a_2,x_1)\beta(a_1)\\
&\quad +\alpha(a_1)J(a_2,x_1)\beta(ra_2a_1)+J(a_1,x_1)\beta(a_2ra_2a_1)-\alpha(a_1a_2r)J(a_2a_1,x_1)\\
&\quad -\alpha(a_2a_1r)J(a_1a_2,x_1)-J(a_1a_2,x_1)\beta(ra_2a_1)-J(a_2a_1,x_1)\beta(ra_1a_2)\\
&= \alpha(a_2a_1r)\big(\alpha(a_1)J(a_2,x_1)+J(a_1,x_1)\beta(a_2)-J(a_1a_2,x_1)\big)\\
&\quad +\big(\alpha(a_1)J(a_2,x_1)+J(a_1,x_1)\beta(a_2)-J(a_1a_2,x_1)\big)\beta(ra_2a_1)\\
&\quad +\alpha(a_1a_2r)\big(\alpha(a_2)J(a_1,x_1)+J(a_2,x_1)\beta(a_1)-J(a_2a_1,x_1)\big)\\
&\quad +\big(\alpha(a_2)J(a_1,x_1)+J(a_2,x_1)\beta(a_1)-J(a_2a_1,x_1)\big)\beta(ra_1a_2)
\end{aligned}
$$

for all $a_1,a_2,x_1\in R$. Using Lemma (2.1), it is easily seen

$$
\begin{aligned}
J(a_1a_2,x_1)-\alpha(a_1)J(a_2,x_1)-J(a_1,x_1)\beta(a_2) &= -J(a_2a_1,x_1)+\alpha(a_2)J(a_1,x_1)\\
&\quad +J(a_2,x_1)\beta(a_1)
\end{aligned}
$$

for all $a_1,a_2,x_1\in R$. Using this equation above relation, we get

$$
\begin{aligned}
0 &= \big(J(a_1a_2,x_1)-\alpha(a_1)J(a_2,x_1)-J(a_1,x_1)\beta(a_2)\big)\alpha(r)\alpha([a_1,a_2])\\
&\quad +\alpha([a_1,a_2])\alpha(r)\big(J(a_1a_2,x_1)-\alpha(a_1)J(a_2,x_1)-J(a_1,x_1)\beta(a_2)\big)
\end{aligned}
$$

for all $a_1,a_2,r,x_1\in R$.

Theorem 3.3 *If R is a semi-prime ring with $charR\neq 2$ and J is a symmetric Jordan $(\alpha,\beta)-$ biderivation, then R is commutative or J is an symmetric $(\alpha,\beta)-$ biderivation.*

Proof. From Lemma (3.2), we have

$$0 = \big(J(a_1a_2,x_1) - \alpha(a_1)J(a_2,x_1) - J(a_1,x_1)\beta(a_2)\big)\alpha(r)\alpha([a_1,a_2])$$

$$+\alpha([a_1,a_2])\alpha(r)\big(J(a_1a_2,x_1) - \alpha(a_1)J(a_2,x_1) - J(a_1,x_1)\beta(a_2)\big)$$

for all $a_1, a_2, r, x_1 \in R$. Using Lemma (2.2), we get

$$\big(J(a_1a_2,x_1) - \alpha(a_1)J(a_2,x_1) - J(a_1,x_1)\beta(a_2)\big)\alpha(r)\alpha[a_1,a_2] = 0 \qquad (3.2)$$

for all $a_1, a_2, r, x_1 \in R$. Replacing a_2 by $a_2 + a_3$, $a_3 \in R$ in Equation (3.2) and using Equation (3.2), we obtain

$$\begin{aligned} 0 = \; & \big(J(a_1a_3,x_1) - \alpha(a_1)J(a_3,x_1) - J(a_1,x_1)\beta(a_3)\big)\alpha(r)\alpha[a_1,a_2] \\ & +\big(J(a_1a_2,x_1) - \alpha(a_1)J(a_2,x_1) - J(a_1,x_1)\beta(a_2)\big)\alpha(r)\alpha[a_1,a_3] \end{aligned} \qquad (3.3)$$

for all $a_1, a_2, a_3, r, x_1 \in R$.

Now, taking $\Big(\big(J(a_1a_2,x_1) - \alpha(a_1)J(a_2,x_1) - J(a_1,x_1)\beta(a_2)\big)\alpha(r)\alpha[a_1,a_3]\Big)\ z$ $(J(a_1a_2,x_1) - \alpha(a_1)\ J(a_2,x_1) - J(a_1,x_1)\ \beta(a_2))\alpha(r)\alpha[a_1,a_3]$ and using Equation (3.3), this relation turns

$\Big(\big(J(a_1a_2,x_1) - \alpha(a_1)J(a_2,x_1) - J(a_1,x_1)\beta(a_2)\big)\alpha(r)\alpha[a_1,a_3]\Big)\ z\ \ (J(a_1a_3,x_1) - \alpha(a_1)\ J(a_3,x_1) - J(a_1,x_1)\beta(a_3))\alpha(r)\alpha[a_1,a_2]$.

Using Equation (3.2) in this relation, we get

$$\begin{aligned} 0 = \; & \big(J(a_1a_2,x_1) - \alpha(a_1)J(a_2,x_1) - J(a_1,x_1)\beta(a_2)\big)\alpha(r)\alpha[a_1,a_3] \\ & z\big(J(a_1a_2,x_1) - \alpha(a_1)J(a_2,x_1) - J(a_1,x_1)\beta(a_2)\big)\alpha(r)\alpha[a_1,a_3] \end{aligned}$$

for all $a_1, a_2, a_3, r, x_1 \in R$. Using semi-primeness of R in above relation, we have

$$\big(J(a_1a_2,x_1) - \alpha(a_1)J(a_2,x_1) - J(a_1,x_1)\beta(a_2)\big)\alpha(r)\alpha[a_1,a_3] = 0 \qquad (3.4)$$

for all $a_1, a_2, a_3, r, x_1 \in R$. Replacing a_1 by $a_1 + a_4$, $a_4 \in R$ in above relation and using and Equation (3.4), we obtain

$$0 = \big(J(a_1a_2, x_1) - \alpha(a_1)J(a_2, x_1) - J(a_1, x_1)\beta(a_2)\big)\alpha(r)\alpha[a_4, a_3] \\ +\big(J(a_4a_2, x_1) - \alpha(a_4)J(a_2, x_1) - J(a_4, x_1)\beta(a_2)\big)\alpha(r)\alpha[a_1, a_3]$$

for all $a_1, a_2, a_3, a_4, r, x_1 \in R$. Hence, applying smilarly method above paragraph and using Equation (3.2), Equation (3.3) and α is automorphism, we have

$$\big(J(a_1a_2, x_1) - \alpha(a_1)J(a_2, x_1) - J(a_1, x_1)\beta(a_2)\big)r[a_4, a_3] = 0 \qquad (3.5)$$

for all $a_1, a_2, a_3, a_4, r, x_1 \in R$. Now, taking $[J(a_1a_2, x_1) - \alpha(a_1)J(a_2, x_1) - J(a_1, x_1)\beta(a_2), a_3]$ r $[J(a_1a_2, x_1) - \alpha(a_1)J(a_2, x_1) - J(a_1, x_1)\beta(a_2), a_3]$ and using commutator properties, this relation turns

$$\big(J(a_1a_2, x_1) - \alpha(a_1)J(a_2, x_1) - J(a_1, x_1)\beta(a_2)\big) \quad a_3r \quad [J(a_1a_2, x_1) - \alpha(a_1)J(a_2, x_1) \\ -J(a_1, x_1)\beta(a_2), a_3] - a_3\big(J(a_1a_2, x_1) - \alpha(a_1)J(a_2, x_1) - \\ J(a_1, x_1)\beta(a_2)\big)r[J(a_1a_2, x_1) - \alpha(a_1)J(a_2, x_1) - J(a_1, x_1)\beta(a_2), a_3].$$

Using Equation (3.5), we get

$$0 = [J(a_1a_2, x_1) - \alpha(a_1)J(a_2, x_1) - J(a_1, x_1)\beta(a_2), a_3] \\ r[J(a_1a_2, x_1) - \alpha(a_1)J(a_2, x_1) - J(a_1, x_1)\beta(a_2), a_3] \text{ for all } a_1, a_2, a_3, r, x_1 \in R.$$

Using semi-primeness of R in above relation, we have

$$[J(a_1a_2, x_1) - \alpha(a_1)J(a_2, x_1) - J(a_1, x_1)\beta(a_2), a_3] = 0 \text{ for all } a_1, a_2, a_3, r, x_1 \in R.$$

So, we get $J(a_1a_2, x_1) - \alpha(a_1)J(a_2, x_1) - J(a_1, x_1)\beta(a_2) \in Z(R)$ for all $a_1, a_2, x_1 \in R$. Hence, from Equation (3.5), we get

$$\big(J(a_1a_2, x_1) - \alpha(a_1)J(a_2, x_1) - J(a_1, x_1)\beta(a_2)\big)[R, R] = 0 \text{ for all } a_1, a_2, x_1 \in R.$$

Using Corollary (2.4), we obtain

$$[R, R] \subset Z(R) \text{or} \big(J(a_1a_2, x_1) - \alpha(a_1)J(a_2, x_1) - J(a_1, x_1)\beta(a_2)\big) = 0$$

for all $a_1, a_2, x_1 \in R$. If $\left(J(a_1a_2, x_1) - \alpha(a_1)J(a_2, x_1) - J(a_1, x_1)\beta(a_2)\right) = 0$ for all $a_1, a_2, x_1 \in R$, then J is an $(\alpha, \beta) -$ biderivation. If $[R, R] \subset Z(R)$, then R commutative from Lemma (2.3).

Corollary 3.4 *If R is a noncommutative semi-prime ring with $charR \neq 2$ and J is a symmetric Jordan $(\alpha, \beta) -$ biderivation, then J is a symmetric $(\alpha, \beta) -$ biderivation.*

Results on Jordan left $(\alpha, \alpha) -$ biderivation

Lemma 4.1 *If R is a ring with $charR \neq 2$ and J is a symmetric Jordan left $(\alpha, \alpha) -$ biderivation, then the following conditions is satisfied.*

i) $J(a_1a_2 + a_2a_1, x_1) = 2\alpha(a_1)J(a_2, x_1) + 2\alpha(a_2)J(a_1, x_1)$ *for all* $a_1, a_2, x_1 \in R$.

ii) $J(a_1a_2a_1, x_1) = \alpha(a_1^2)J(a_2, x_1) + 3\alpha(a_1a_2)J(a_1, x_1) - \alpha(a_2a_1)J(a_1, x_1)$ *for all* $a_1, a_2, x_1 \in R$.

iii) $J(a_1a_2a_3 + a_3a_2a_1, x_1) = \alpha(a_1a_3 + a_3a_1)J(a_2, x_1) + 3\alpha(a_1a_2)J(a_3, x_1) + 3\alpha(a_3a_2)J(a_1, x_1) - \alpha(a_2a_1)J(a_3, x_1) - \alpha(a_2a_3)J(a_1, x_1)$ *for all* $a_1, a_2, a_3, x_1 \in R$.

Proof. i) Let

$$J(a_1^2, x_1) = 2\alpha(a_1)J(a_1, x_1) \text{ for all } a_1, x_1 \in R. \quad (4.1)$$

Replacing a_1 by $a_1 + a_2$, $a_2 \in R$ in above relation, we have

$$J((a_1 + a_2)^2, x_1) = 2\alpha(a_1 + a_2)J(a_1 + a_2, x_1)$$

$$= 2\alpha(a_1)J(a_2, x_1) + 2\alpha(a_2)J(a_1, x_1)$$

$$+2\alpha(a_1)J(a_1, x_1) + 2\alpha(a_2)J(a_2, x_1)$$

for all $a_1, a_2, x_1 \in R$. Using $J((a_1 + a_2)^2, x_1) = J(a_1^2 + a_1a_2 + a_2a_1 + a_2^2, x_1) = J(a_1^2, x_1) + J(a_1a_2 + a_2a_1) + J(a_2^2, x_1)$ and Equation (4.1) in this relation, we get

$$J(a_1a_2 + a_2a_1, x_1) = 2\alpha(a_1)J(a_2, x_1) + 2\alpha(a_2)J(a_1, x_1) \text{ for all } a_1, a_2, x_1 \in R.$$

$ii)$ Replacing a_2 by $a_2 a_1$ in (i), we get

$$J(a_1(a_2a_1)+(a_2a_1)a_1,x_1)=2\alpha(a_1)J(a_2a_1,x_1)+2\alpha(a_2a_1)J(a_1,x_1)$$

for all $a_1,a_2,x_1\in R$. Also, replacing a_2 by a_1a_2 in (i), we obtain

$$J(a_1(a_1a_2)+(a_1a_2)a_1,x_1)=2\alpha(a_1)J(a_1a_2,x_1)+2\alpha(a_1a_2)J(a_1,x_1)$$

for all $a_1,a_2,x_1\in R$.Summationing this relations and using $J(a_1(a_2a_1)+(a_2a_1)a_1,x_1)=J(a_1a_2a_1+a_2a_1^2,x_1)=J(a_1a_2a_1,x_1)+J(a_2a_1^2,x_1)$ and $J(a_1(a_1a_2)+(a_1a_2)a_1,x_1)=J(a_1^2a_2+a_1a_2a_1,x_1)=J(a_1^2a_2,x_1)+J(a_1a_2a_1,x_1)$, we have

$$2J(a_1a_2a_1,x_1)=2\alpha(a_1)J(a_2a_1,x_1)+2\alpha(a_2a_1)J(a_1,x_1)+2\alpha(a_1)J(a_1a_2,x_1)$$
$$+2\alpha(a_1a_2)J(a_1,x_1)-J(a_1^2a_2+a_2a_1^2,x_1) \tag{4.2}$$

for all $a_1,a_2,x_1\in R$. Using (i) for expression $-J(a_1^2a_2+a_2a_1^2,x_1)$, we have

$J(a_1^2a_2+a_2a_1^2,x_1)=2\alpha(a_1^2)J(a_2,x_1)+2\alpha(a_2)J(a_1^2,x_1)$, for all $a_1,a_2,x_1\in R$.

Using this relation in Equation (4.2), we get

$$2J(a_1a_2a_1,x_1)=2\alpha(a_1)J(a_1a_2+a_2a_1,x_1)+2\alpha(a_2a_1)J(a_1,x_1)$$
$$+2\alpha(a_1a_2)J(a_1,x_1)-2\alpha(a_1^2)J(a_2,x_1)-2\alpha(a_2)2\alpha(a_1)J(a_1,x_1)$$

for all $a_1,a_2,x_1\in R$. Using (i) for expression $J(a_1a_2+a_2a_1,x_1)$, we have

$$2J(a_1a_2a_1,x_1) = 2\alpha(a_1^2)J(a_2,x_1)+6\alpha(a_1a_2)J(a_1,x_1)-2\alpha(a_2a_1)J(a_1,x_1)$$

for all $a_1,a_2,x_1\in R$. From $charR\neq 2$, for all $a_1,a_2,x_1\in R$, we get

$$J(a_1a_2a_1,x_1) = \alpha(a_1^2)J(a_2,x_1)+3\alpha(a_1a_2)J(a_1,x_1)-\alpha(a_2a_1)J(a_1,x_1)$$

$iii)$ Replacing a_1 by a_1+a_3, $a_3\in R$ in (ii), we get

$$J\big((a_1+a_3)a_2(a_1+a_3),x_1\big)=\alpha((a_1+a_3)^2)J(a_2,x_1)+3\alpha\big((a_1+a_3)a_2\big)$$
$$J\big((a_1+a_3),x_1\big)-\alpha\big(a_2(a_1+a_3)\big)J\big((a_1+a_3),x_1\big)$$

for all $a_1,a_2,x_1\in R$. Using $J\big((a_1+a_3)a_2(a_1+a_3),x_1\big)=J(a_1a_2a_1+a_1a_2a_3+a_3a_2a_1+a_3a_2a_3,x_1)$, we obtain

$$J(a_1a_2a_3+a_3a_2a_1,x_1)=\alpha(a_1^2)J(a_2,x_1)+\alpha(a_1a_3+a_3a_1)J(a_2,x_1)$$
$$+\alpha(a_3^2)J(a_2,x_1)+3\alpha(a_1a_2)J(a_1,x_1)+3\alpha(a_1a_2)J(a_3,x_1)$$
$$+3\alpha(a_3a_2)J(a_1,x_1)+3\alpha(a_3a_2)J(a_3,x_1)-\alpha(a_2a_1)J(a_1,x_1)$$
$$-\alpha(a_2a_1)J(a_3,x_1)-\alpha(a_2a_3)J(a_3,x_1)-\alpha(a_2a_3)J(a_1,x_1)$$
$$-J(a_1a_2a_1,x_1)-J(a_3a_2a_3,x_1)$$

for all $a_1,a_2,x_1\in R$. Using (ii) for expressions $J(a_1a_2a_1,x_1)$ and $J(a_3a_2a_3,x_1)$in this relation, for all $a_1,a_2,a_3,x_1\in R$, we have

$$J(a_1a_2a_3+a_3a_2a_1,x_1) \quad = \quad \alpha(a_1a_3+a_3a_1)J(a_2,x_1)+3\alpha(a_1a_2)J(a_3,x_1)$$
$$+3\alpha(a_3a_2)J(a_1,x_1)-\alpha(a_2a_1)J(a_3,x_1)-\alpha(a_2a_3)J(a_1,x_1)$$

Lemma 4.2 *If R is a prime ring with $charR\neq 2,3$, J is a symmetric Jordan left $(\alpha,\alpha)-$ biderivation and $a_1\in R$, then the following statements are satisfied.*

i) If $J(a_1,x_1)\neq 0$ for some $x_1\in R$, then $\big[a_1,[a_1,a_2]\big]^2=0$ for all $a_2\in R$.

ii) If $a_1^2=0$, then $J(a_1,x_1)=0$ for all $x_1\in R$.

Proof. i) Let $J(a_1,x_1)\neq 0$ for $x_1\in R$. Replacing a_3 by a_1a_2 in Lemma 4.1 (iii), we get

$$J(a_1a_2(a_1a_2)+(a_1a_2)a_2a_1,x_1)=\alpha(a_1(a_1a_2)+(a_1a_2)a_1)J(a_2,x_1)$$
$$+3\alpha(a_1a_2)J\big((a_1a_2),x_1\big)+3\alpha\big((a_1a_2)a_2\big)J(a_1,x_1)$$

$$-\alpha(a_2a_1)J\big((a_1a_2),x_1\big)-\alpha\big(a_2(a_1a_2)\big)J(a_1,x_1)$$

for all $a_2,x_1\in R$. Also, using $J(a_1a_2(a_1a_2)+(a_1a_2)a_2a_1,x_1)=J((a_1a_2)^2+a_1a_2^2a_1,x_1)$ and Lemma 4.1 (ii), we obtain

$$\begin{aligned} J((a_1a_2)^2+a_1a_2^2a_1,x_1) &= 2\alpha(a_1a_2)J(a_1a_2,x_1)+\alpha(a_1^2)J(a_2^2,x_1)\\ &\quad +3\alpha(a_1a_2^2)J(a_1,x_1)-\alpha(a_2^2a_1)J(a_1,x_1)\end{aligned}$$

for all $a_2,x_1\in R$. Using equality of above expressions, we get

$$\begin{aligned} 0 &= -\alpha(a_1^2a_2)J(a_2,x_1)+\alpha(a_1a_2a_1)J(a_2,x_1)+\alpha(a_1a_2)J(a_1a_2,x_1)\\ &\quad -\alpha(a_2a_1)J(a_1a_2,x_1)-\alpha(a_2a_1a_2)J(a_1,x_1)+\alpha(a_2^2a_1)J(a_1,x_1)\end{aligned} \tag{4.3}$$

for all $a_2,x_1\in R$. Replacing a_2 by a_1+a_2 in Equation (4.3), we have

$$\begin{aligned} 0 &= -2\alpha(a_1^2a_2)J(a_1,x_1)+4\alpha(a_1a_2a_1)J(a_1,x_1)-2\alpha(a_2a_1^2)J(a_1,x_1)\\ &\quad -\alpha(a_1^2a_2)J(a_2,x_1)+\alpha(a_1a_2a_1)J(a_2,x_1)+\alpha(a_1a_2)J(a_1a_2,x_1)\\ &\quad -\alpha(a_2a_1)J(a_1a_2,x_1)-\alpha(a_2a_1a_2)J(a_1,x_1)+\alpha(a_2^2a_1)J(a_1,x_1)\end{aligned}$$

for all $a_2,x_1\in R$. Using Equation (4.3) and $charR\neq 2$, we obtain

$$-\alpha(a_1^2a_2)J(a_1,x_1)+2\alpha(a_1a_2a_1)J(a_1,x_1)-\alpha(a_2a_1^2)J(a_1,x_1)=0 \text{ for all } a_2,x_1\in R.$$

If above expression is rearranged, we get

$$\alpha\big[a_1,[a_1,a_2]\big]J(a_1,x_1)=0 \text{ for all } a_2,x_1\in R. \tag{4.4}$$

Replacing a_2 by a_2a_3, $a_3\in R$ in above relation and using Equation (4.4), we have

$$\big(2\alpha[a_1,a_2][a_1,a_3]+\alpha\big[a_1,[a_1,a_2]\big]a_3\big)J(a_1,x_1)=0 \text{ for all } a_2,a_3,x_1\in R.$$

Replacing a_3 by $[a_1,a_3a_4]$, $a_4\in R$ in above relation and using Equation (4.4), we have

$$\alpha\big(\big[a_1,[a_1,a_2]\big][a_1,a_3]a_4\big)J(a_1,x_1)+\alpha\big(\big[a_1,[a_1,a_2]\big]a_3[a_1,a_4]\big)J(a_1,x_1)=0 \tag{4.5}$$

for all $a_2,a_3,a_4,x_1\in R$. Replacing a_4 by $[a_1,a_4]$ in above relation and using Equation (4.4), we obtain

$$\alpha\big([a_1,[a_1,a_2]][a_1,a_3][a_1,a_4]\big)J(a_1,x_1)=0 \qquad (4.6)$$

for all $a_2,a_3,a_4,x_1\in R$. Now, replacing a_3 by $[a_1,a_3]$ in Equation (4.5) and using Equation (4.6), we get

$$\alpha\big([a_1,[a_1,a_2]][a_1,[a_1,a_3]]\big)\alpha(a_4)J(a_1,x_1)=0$$

for all $a_2,a_3,a_4,x_1\in R$. Using primeness of R, $J(a_1,x_1)\neq 0$ and α is automorphism in above relation, we have

$[a_1,[a_1,a_2]][a_1,[a_1,a_3]]=0$ for all $a_2,a_3\in R$.

So, for $a_3=a_2$, we get $[a_1,[a_1,a_2]]^2=0$ for all $a_2\in R$.

ii) Let $a_1^2=0$. Using $0=J(a_1^2,x_1)=2\alpha(a_1)J(a_1,x_1)$ and $charR\neq 2$, we get

$$\alpha(a_1)J(a_1,x_1)=0 \text{ for all } x_1\in R. \qquad (4.7)$$

Now, taking $a_1a_2a_1a_3a_1+a_1a_3a_1a_2a_1\in R$, $a_2,a_3\in R$ and using Lemma 4.1 (ii) and Equation (4.7) , we have

$$J(a_1(a_2a_1a_3+a_3a_1a_2)a_1,x_1) \quad = \quad 3\alpha(a_1a_2a_1a_3+a_1a_3a_1a_2)J(a_1,x_1)$$

for all $a_2,a_3,x_1\in R$. Also, using Lemma 4.1 (iii), Equation (4.7) and $a_1^2=0$, we get

$$\begin{aligned} J(a_1a_2(a_1a_3a_1)+(a_1a_3a_1)a_2a_1,x_1) \quad &= \quad 9\alpha(a_1a_2a_1a_3)J(a_1,x_1) \\ &\quad +3\alpha(a_1a_3a_1a_2)J(a_1,x_1) \end{aligned}$$

for all $a_2,a_3,x_1\in R$. From equality of two expressions, for all $a_2,a_3,x_1\in R$, we obtain $6\alpha(a_1a_2a_1a_3)J(a_1,x_1)=0$. Using $charR\neq 2,3$, α is automorphism, for all $a_2,x_1\in R$ we have $a_1a_2a_1=0$ or $J(a_1,x_1)=0$. Using primeness of R in this relation, we obtain

$a_1=0$ or $J(a_1,x_1)=0$ for all $x_1\in R$. In both cases, $J(a_1,x_1)=0$ is hold for all $x_1\in R$.

Theorem 4.3 *If R is a prime ring and $charR\neq 2,3$, J is a nonzero symmetric Jordan left $(\alpha,\alpha)-$ biderivation, then R is commutative.*

Proof. Let $J(a_1, r) \neq 0$ for $a_1, r \in R$. Then $\left[a_1, [a_1, x_1]\right]^2 = 0$ for all $x_1 \in R$ from Lemma 4.2 (i). Hence, $J\left(\left[a_1, [a_1, x_1]\right], r\right) = 0$ for all $x_1 \in R$. Using $J\left(\left[a_1, [a_1, x_1]\right], r\right) = J(a_1^2 x_1 + x_1 a_1^2, r) - 2J(a_1 x_1 a_1, r)$, we get $6\alpha[a_1, x_1]J(a_1, r) = 0$ for all $x_1 \in R$. Using $charR \neq 2,3$, we have

$$\alpha[a_1, x_1]J(a_1, r) = 0 \text{ for all } x_1 \in R. \tag{4.8}$$

Replacing x_1 by $x_1 x_2$, $x_2 \in R$ in above relation, then using Equation (4.8), we obtain $\alpha[a_1, x_1]x_2 J(a_1, r) = 0$ for all $x_1, x_2 \in R$. Using primeness of R in this relation, we have $\alpha[a_1, x_1] = 0$ or $J(a_1, r) = 0$ for all $x_1 \in R$. Using $J(a_1, r) \neq 0$ from assumption and α is automorphism, we get $a_1 \in Z(R)$. So, if there exist $r \in R$ such that $J(a_1, r) \neq 0$, then $a_1 \in Z(R)$. Hence, we have $a_1 \in Z(R)$ or $J(a_1, x_1) = 0$ for all $x_1 \in R$.

Let $C = \{a_1 \in R | a_1 \in Z(R)\}$ and $E = \{a_1 \in R | J(a_1, x_1) = 0 \text{ for all } x_1 \in R.\}$. C and E are subgroups of additive group R whose $R = C \cup E$, but R can't be written as a union of its two proper subgroups. Hence, $R = C$ or $R = E$. Since $J \neq 0$ by hypothesis, $R = C$ and R is commutative.

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