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Euclidean Curves with Incompressible Canonical Vector Field

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Abstract

In the present study we consider Euclidean curves with incompressible canonical vector fields. We investigate such curves in terms of their curvature functions. Recently, B.Y. Chen gave classification of plane curves with incompressible canonical vector fields. For higher dimensional case we gave a complete classification of Euclidean space curves with incompressible canonical vector fields. Further we obtain some results of the Euclidean curves with incompressible canonical vector fields in 4-dimensional Euclidean space $\mathbb{E}^4$.

Keywords: Regular curve, Generalized helix, Salkowski curve, Canonical vector field.

Sıkıştırılamayan Kanonik Vektör Alana Sahip Öklit Eğrileri

Özet

Bu çalışmada Öklit uzayında sıkıştırılamayan kanonik vektör alana sahip eğriler ele alınmıştır. Bu tür eğrilerin eğrilik fonksiyonları incelenmiştir. Son zamanlarda B.Y. Chen sıkıştırılamayan kanonik vektör alana sahip düzlemsel eğrilerin bir sınıflandırmasını vermiştir. Bu çalışmada yüksek boyutlu Öklit uzayında bu tür eğrilerin

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bir sınıflandırması verilmiştir. Özellikle 4-boyutlu Öklit uzayında bazı sonuçlar elde edilmiştir.

Anahtar Kelimeler: Regüler eğri, Genelleştirilmiş helis, Salkowski eğrisi, Kanonik vektör alanı.

1. Introduction

Let $\alpha = \alpha(t) : I \subset \mathbb{R} \to \mathbb{E}^m$ be a regular curve in $\mathbb{E}^m$, (i.e., $\left\| \alpha^{'}(t) \right\| \neq 0$). Then α is called a *Frenet curve of osculating order* d, $(2 \leq d \leq m)$ if $\alpha^{'}(t), \alpha^{''}(t), ..., \alpha^{(d)}(t)$ are linearly independent and $\alpha^{'}(t), \alpha^{''}(t), ..., \alpha^{(d+1)}(t)$ linearly dependent for all t in I [10]. In this case, $Im(\alpha)$ lies in a $d-$dimensional Euclidean subspace of $\mathbb{E}^m$. To each Frenet curve of osculating order d there can be associated orthonormal $d-$frame $V_1 = \dfrac{\alpha^{'}(t)}{\left\| \alpha^{'}(t) \right\|}, ..., V_d$ along α, the Frenet $d-$frame, and $d-1$ functions $\kappa_1, \kappa_2, ..., \kappa_{d-1} : I \to \mathbb{R}$, the *Frenet curvatures*, such that

$$\begin{bmatrix} V_1^{'} \\ V_2^{'} \\ V_3^{'} \\ ... \\ V_d^{'} \end{bmatrix} = v \begin{bmatrix} 0 & \kappa_1 & 0 & ... & 0 \\ -\kappa_1 & 0 & \kappa_2 & ... & 0 \\ 0 & -\kappa_2 & 0 & ... & 0 \\ ... & & & & \kappa_{d-1} \\ 0 & 0 & ... & -\kappa_{d-1} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_3 \\ V_3 \\ ... \\ V_d \end{bmatrix} \tag{1}$$

where, $v = \left\| \alpha^{'}(t) \right\|$ is the speed of the curve α. In fact, to obtain $V_1, ..., V_d$ it is sufficient to apply the Gram-Schmidt orthonormalization process to $\alpha^{'}(t), \alpha^{''}(t), ..., \alpha^{(d)}(t)$. Moreover, the functions $\kappa_1, \kappa_2, ..., \kappa_{d-1}$ are easily obtained by using the above Frenet equations. More precisely, $V_1, ..., V_d$ and $\kappa_1, \kappa_2, ..., \kappa_{d-1}$ are determined by the following formulas:

$$E_1(t) := \alpha^{'}(t) \quad ; V_1 := \frac{E_1(t)}{\left\| E_1(t) \right\|},$$

$$E_\sigma(t) := \alpha^{(\sigma)}(t) - \sum_{i=1}^{\sigma} < \alpha^{(\sigma)}(t), E_i(t) > \frac{E_i(t)}{\left\| E_i(t) \right\|^2},$$

$$V_\sigma := \frac{E_\sigma(t)}{\|E_\sigma(t)\|},$$

$$\kappa_\delta(s) := \frac{\|E_{\delta+1}(t)\|}{\|E_\delta(t)\| \|E_1(t)\|},$$

where $\delta \in \{1,2,3,...,d-1\}$ (see, [4]). For the case $d = n$ the Frenet curve α is called a *generic curve* [3, 10]. A Frenet curve of rank d for which $\kappa_1, \kappa_2, ..., \kappa_{d-1}$ are constant is called *(generalized) helix* or $W-$*curve* [5]. Meanwhile, a Frenet curve of rank d with constant curvature ratios $\frac{\kappa_2}{\kappa_1}, \frac{\kappa_3}{\kappa_2}, \frac{\kappa_4}{\kappa_3}, ..., \frac{\kappa_{d-1}}{\kappa_{d-2}}$ is called a $ccr-$*curve* [6, 7].

A generic space curve with constant first curvature κ_1 and non-constant second curvature κ_2 is called a *Salkowski curve* [1, 9]. A generic curve in $\mathbb{E}^4$ is called a *slope curve* for which the curvatures $\kappa_1 \neq 0$, κ_2 and κ_3 satisfy the relations

$$\frac{\kappa_2}{\kappa_1} = \lambda, \frac{\kappa_3}{\kappa_1} = \mu, \tag{2}$$

where λ and μ are non-zero real constants [5].

2. Euclidean Curves with Incompressible Canonical Vector Field

Let $\alpha = \alpha(s) : I \subset \mathbb{R} \to \mathbb{E}^m$ be a regular curve in $\mathbb{E}^m$ given with the arclength parameter s. For the Euclidean curve $\alpha(s)$ there exists a natural decomposition of the position vector field α given by:

$$\alpha(s) = \alpha(s)^{\mathrm{T}} + \alpha(s)^{\perp}, \tag{3}$$

where $\alpha(s)^{\mathrm{T}}$ and $\alpha(s)^{\perp}$ denote the tangential and the normal components of α, respectively.

A vector field υ on a Riemannian manifold (M, g) is called *conservative* if it is the gradient of some function, known as a scalar potential. Conservative vector fields appear naturally in mechanics: They are vector fields representing forces of physical systems in which energy is conserved [2].

For Euclidean curves we give the following definition.

Definition 2.1 *Let* $\alpha = \alpha(s): I \subset \mathbb{R} \to \mathbb{E}^m$ *be a unit speed, regular curve in* $\mathbb{E}^m$. *If the divergence of the the canonical vector field* $\alpha(s)^{\mathrm{T}}$ *of* α *vanishes identicall. That is; if*

$$div(\alpha^{\mathrm{T}}(s)) = 0,$$

holds then the vector field $\alpha(s)^{\mathrm{T}}$ *is called incompressible.*

Using a result of B.Y. Chen in [2] one can get the following adapted result.

Theorem 2.2 *Let* $\alpha = \alpha(s): I \subset \mathbb{R} \to \mathbb{E}^m$ *be a unit speed, regular curve in* $\mathbb{E}^m$. *Then the canonical vector field* $\alpha(s)^{\mathrm{T}}$ *of* α *is incompressible if and only if*

$$\langle \Delta\alpha(s), \alpha(s) \rangle = 1, \tag{4}$$

holds, where $\Delta = -\dfrac{d^2}{ds^2}$ *is the Laplacian of* α.

Proof. The divergence of the canonical vector field $\alpha(s)^{\mathrm{T}}$ of α is given by

$$div(\alpha(s)^T) = \left\langle \widetilde{\nabla}_{\alpha'(s)}\alpha(s)^T, \alpha'(s) \right\rangle \tag{5}$$

where $\widetilde{\nabla}$ is the covariant derivative in $\mathbb{E}^m$. For the scalar potential function

$$f = \frac{1}{2}\langle \alpha, \alpha \rangle,$$

the gradient of f becomes

$$\begin{aligned} grad(f) &= \frac{1}{2}\widetilde{\nabla}_{\alpha'(s)}\langle \alpha(s), \alpha(s)\rangle \alpha'(s) \\ &= \langle \alpha'(s), \alpha(s)\rangle \alpha'(s) = \alpha(s)^T \end{aligned} \tag{6}$$

which means that, the canonical vector field $\alpha(s)^{\mathrm{T}}$ is conservative. Hence, substituting (6) into (5) after some computation we get

$$
\begin{aligned}
div(\alpha(s)^T) &= \left\langle \widetilde{\nabla}_{\alpha'(s)}\alpha(s)^T, \alpha'(s) \right\rangle \\
&= \left\langle \widetilde{\nabla}_{\alpha'(s)} \left\{ \left\langle \alpha'(s), \alpha(s) \right\rangle \alpha'(s) \right\}, \alpha'(s) \right\rangle \\
&= \left\langle \widetilde{\nabla}_{\alpha'(s)}\alpha'(s), \alpha(s) \right\rangle \left\langle \alpha'(s), \alpha'(s) \right\rangle \\
&+ \left\langle \alpha'(s), \widetilde{\nabla}_{\alpha'(s)}\alpha(s) \right\rangle \left\langle \alpha'(s), \alpha'(s) \right\rangle \\
&+ \left\langle \alpha'(s), \alpha(s) \right\rangle \left\langle \widetilde{\nabla}_{\alpha'(s)}\alpha'(s), \alpha'(s) \right\rangle
\end{aligned}
$$

Furthermore, using the equalities

$$
\begin{aligned}
&\left\langle \widetilde{\nabla}_{\alpha'(s)}\alpha'(s), \alpha'(s) \right\rangle = 0, \\
&\left\langle \alpha'(s), \alpha'(s) \right\rangle = 1, \quad \widetilde{\nabla}_{\alpha'(s)}\alpha(s) = \alpha'(s)
\end{aligned}
$$

into previous equation one can get

$$
div(\alpha(s)^T) = \left\langle \widetilde{\nabla}_{\alpha'(s)}\alpha'(s), \alpha(s) \right\rangle + 1
$$

If the canonical vector field $\alpha(s)^{\mathrm{T}}$ of α is incompressible then by definition $div(\alpha(s)^{\mathrm{T}}) = 0$ holds identically. So, we obtain

$$
\left\langle \widetilde{\nabla}_{\alpha'(s)}\alpha'(s), \alpha(s) \right\rangle = -1 \tag{7}
$$

which gives the proof of the theorem.

2.1 Planar Curves

Let $\alpha = \alpha(t) : I \subset \mathbb{R} \to \mathbb{E}^2$ be a unit speed regular curve in $\mathbb{E}^2$. Then one can get the following Frenet equations;

$$
\begin{aligned}
&\alpha'(s) = T(s), \\
&\alpha''(s) = T'(s) = \kappa N(s), \\
&N'(s) = -\kappa T(s),
\end{aligned}
$$

where $\{T, N\}$ is the Frenet frame of α and $\kappa > 0$ is the curvature (function) of α.

For the plane curves we have the following classification theorem of B.Y. Chen;

Theorem 2.3 [2] *Let* $\alpha = \alpha(s): I \subset \mathbb{R} \to \mathbb{E}^2$ *be a unit speed, regular curve in* $\mathbb{E}^2$. *Then the canonical vector field* $\alpha(s)^{\mathrm{T}}$ *of* α *is incompressible if and only if up to a rigid motion in* $\mathbb{E}^2$ *about the origin,* α *is an open portion of a curve of the following types;*

a) A circle centered at the origin,

b) A curve defined by the parametrization

$$\alpha(s) = \frac{2}{c^2}\left(\cos\left(c\sqrt{s}\right) + c\sqrt{s}\sin\left(c\sqrt{s}\right), \sin\left(c\sqrt{s}\right) - c\sqrt{s}\cos\left(c\sqrt{s}\right)\right),$$

for some non-zero real number c.

2.2 Space Curves

Let $\alpha = \alpha(s)$ be a regular space curve in $\mathbb{E}^3$ given with the arclength parameter s. Then we have the following Frenet equations;

$$\begin{aligned}
&\alpha'(s) = T(s),\\
&\alpha''(s) = T'(s) = \kappa_1 N(s),\\
&N'(s) = -\kappa_1 T(s) + \kappa_2 B(s),\\
&B'(s) = -\kappa_2 N(s),
\end{aligned} \tag{8}$$

where T, N and B are the Frenet frame fields of α and $\kappa_1 > 0$ and κ_2 are the curvature functions of α.

From now on let us assume that α is a space curve whose canonical vector field $\alpha(s)^{\mathrm{T}}$ is incompressible. Then it follows from (7) and (8) that

$$\langle N(s), \alpha(s) \rangle = -\frac{1}{\kappa_1}. \tag{9}$$

Differentiating (9) with respect to s and using (8) we have

$$-\kappa_1 \langle T(s), \alpha(s) \rangle + \kappa_2 \langle B(s), \alpha(s) \rangle = \frac{\kappa_1'}{\kappa_1^2}. \tag{10}$$

Similarly, differentiating (10) with respect to s and using the Frenet equations (8) we get

$$-\kappa_1^{'}\left\langle T(s),\alpha(s)\right\rangle+\kappa_2^{'}\left\langle B(s),\alpha(s)\right\rangle=\left(\frac{\kappa_1^{'}}{\kappa_1^2}\right)^{'}-\frac{\kappa_2^2}{\kappa_1}. \quad (11)$$

Consequently, the equations (10) and (11) have a common solution

$$h\left\langle T(s),\alpha(s)\right\rangle=\frac{\kappa_2^3}{\kappa_1}+g, \quad (12)$$

$$h\left\langle B(s),\alpha(s)\right\rangle=\kappa_2^2+f, \quad (13)$$

where f,g and h are smooth functions defined by

$$\varphi=\frac{\kappa_1^{'}}{\kappa_1^2},$$

$$f=\kappa_1^{'}\varphi-\kappa_1\varphi^{'}, \quad (14)$$

$$g=\kappa_2^{'}\varphi-\kappa_2\varphi^{'},$$

$$h=\kappa_1^{'}\kappa_2-\kappa_2^{'}\kappa_1,$$

respectively. So, we have the following cases;

Case (a): Suppose that If $h=0$ holds. In this case (14) implies that the equality $\kappa_1\kappa_2^{'}-\kappa_1^{'}\kappa_2=0$ holds identically. Thus, the ratio $\frac{\kappa_2}{\kappa_1}=\lambda$ is a real constant. So, the curve $\alpha=\alpha(s)$ is a cylindrical helix. We have the following subcases;

($\mathbf{a}_1$): If κ_1 and κ_2 are both constant curvatures, i.e., α is a $W-$curve. In this case $f=0$ and $g=0$ holds identically. Thus, (12) and (13) gives $\kappa_2=0$. Hence $\alpha=\alpha(s)$ is an open portion of a circle centered at the origin.

($\mathbf{a}_2$): If κ_1 and κ_2 are both non-constant curvatures. Then, differentiating the equations (12) and (13) with respect to s and using (14) with $\kappa_2=\lambda\kappa_1$ we obtain the following differential equation

$$3\left(\kappa_1^{'}\right)^2-\kappa_1^{''}\kappa_1+\lambda^2\kappa_1^4=0, \quad (15)$$

where λ is a non-zero constant function. A simple calculation gives that the differential equation (15) has a non-trivial solution

$$\kappa_1 = \pm \frac{1}{\sqrt{-\lambda^2 s^2 + 2as - 2b}}, \tag{16}$$

for some real numbers a, b and λ.

Summing up the equalities above we obtain the following result;

Theorem 2.4 *Let $\alpha = \alpha(s) : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed, regular curve in $\mathbb{E}^3$ given with incompressible canonical vector field $\alpha(s)^{\mathrm{T}}$. If α is a helical curve then α is either an open portion of a circle centered at the origin or a cylindrical helix given with the Frenet curvatures*

$$\kappa_1 = \pm \frac{1}{\sqrt{-\lambda^2 s^2 + 2as - 2b}}, \kappa_2 = \lambda \kappa_1,$$

where a, b and λ are real constants.

Case (b): Suppose that $h \neq 0$ holds. In this case (12) and (13) gives

$$\langle T(s), \alpha(s) \rangle = \frac{\kappa_2^3 + \kappa_1 g}{\kappa_1 h}, \tag{17}$$

$$\langle B(s), \alpha(s) \rangle = \frac{\kappa_2^2 + f}{h}. \tag{18}$$

Differentiating both equations (17) and (18) with respect to s and using the Frenet formulae (8) we obtain the following equations

$$0 = \left(\frac{\kappa_2^2 + \kappa_1 g}{\kappa_1 h} \right)' \tag{19}$$

$$\frac{\kappa_2}{\kappa_1} = \left(\frac{\kappa_2^2 + f}{h} \right)', \tag{20}$$

respectively.

Summing up the equalities (17)-(20) we obtain the following result.

Theorem 2.5 *Let* $\alpha = \alpha(s): I \subset \mathbb{R} \to \mathbb{E}^3$ *be a unit speed, non-helical curve in* $\mathbb{E}^3$. *Then the canonical vector field* $\alpha(s)^{\mathrm{T}}$ *of* α *is incompressible if and only if the following two equalities hold;*

$$0 = \left(\frac{\kappa_2^3 + \kappa_1 g}{\kappa_1 h} \right)',$$

$$\frac{\kappa_2}{\kappa_1} = \left(\frac{\kappa_2^2 + f}{h} \right)',$$

where

$$\varphi = \frac{\kappa_1'}{\kappa_1^2},$$

$$f = \kappa_1' \varphi - \kappa_1 \varphi',$$

$$g = \kappa_2' \varphi - \kappa_2 \varphi',$$

$$h = \kappa_1' \kappa_2 - \kappa_2' \kappa_1,$$

are real valued smooth functions with $h \neq 0$.

As a consequence of Theorem 2.5 we obtain the following results;

Corollary 2.6 *Let* $\alpha = \alpha(s): I \subset \mathbb{R} \to \mathbb{E}^3$ *be a unit speed, non-helical curve in* $\mathbb{E}^3$. *If* α *is a Salkowski curve with the incompressible canonical vector field* $\alpha(s)^{\mathrm{T}}$ *then*

$$\kappa_2 = \pm \frac{1}{\sqrt{-2cs - 2d}}, \tag{21}$$

holds identically.

Proof. Let α be a Salkowski curve with the incompressible canonical vector field $\alpha(s)^{\mathrm{T}}$. Then using (14) we get

$$f = g = 0, h = -\kappa_1 \kappa_2',$$

where κ_1 is a real constant. Consequently, substituting these values into (19) and (20) we get the same differential equation

$$3\left(\kappa_2'\right)^2 - \kappa_2''\kappa_2 = 0, \tag{22}$$

for the both equations (19) and (20) respectively. An easy calculation gives that the differential equation (22) has a non-trivial solution (21).

2.3 Curves in Euclidean 4-space $\mathbb{E}^4$

Let $\alpha = \alpha(s)$ be a regular curve in $\mathbb{E}^4$ given with the arclength parameter s. Then we have the following Frenet equations;

$$\begin{aligned}
&\alpha'(s) = T(s),\\
&\alpha''(s) = T'(s) = \kappa_1(s)N(s),\\
&N'(s) = -\kappa_1(s)T(s) + \kappa_2(s)B(s),\\
&B'(s) = -\kappa_2(s)N(s) + \kappa_3(s)W(s),\\
&W'(s) = -\kappa_3(s)B(s),
\end{aligned} \tag{23}$$

where T, N, B and W are the Frenet frame fields of α and $\kappa_1 > 0,$ κ_2 and κ_3 are the curvature functions.

It is well-known that the regular parametric curve α in $\mathbb{E}^4$ has the position vector field of the form

$$\alpha(s) = m_0(s)T(s) + m_1(s)N(s) + m_2(s)B(s) + m_3(s)W(s), \tag{24}$$

where $m_i(s), 0 \le i \le 3$ are differentiable functions and T, N, B and W are the Frenet frame fields of α. Differentiating (24) with respect to arclength parameter s and using the Frenet equations (23), we obtain

$$\begin{aligned}
\alpha'(s) &= (m_0'(s) - \kappa_1(s)m_1(s))T(s)\\
&+(m_1'(s) + \kappa_1(s)m_0(s) - \kappa_2(s)m_2(s))N(s)\\
&+(m_2'(s) + \kappa_2(s)m_1(s) - \kappa_3(s)m_3(s))B(s)\\
&+(m_3'(s) + \kappa_3(s)m_2(s))W,
\end{aligned} \tag{25}$$

which follows

$$m_0' - \kappa_1 m_1 = 1,$$

$$m_1' + \kappa_1 m_0 - \kappa_2 m_2 = 0, \tag{26}$$

$$m_2' + \kappa_2 m_1 - \kappa_3 m_3 = 0,$$

$$m_3' + \kappa_3 m_2 = 0,$$

(see, [8]).

From now on let us assume that α is a space curve whose canonical vector field $\alpha(s)^{\mathrm{T}}$ is incompressible. Then it follows from (7) and (23) that

$$m_1 = \langle N(s), \alpha(s) \rangle = -\frac{1}{\kappa_1}, \tag{27}$$

holds. So, from the first equation of (26) we deduce that

$$m_0 = \langle T(s), \alpha(s) \rangle, \tag{28}$$

is a constant function. Further, the equations in (26) imply that

$$m_2 = \frac{\kappa_1' + \kappa_1^3 m_0}{\kappa_1^2 \kappa_2},$$

$$m_3 = \frac{m_2' + \kappa_2 m_1}{\kappa_3}, \tag{29}$$

$$m_3' = -\kappa_3 m_2,$$

holds identically.

In [8] the first author and at all. gave the following definition;

Definition 2.7 *Let $\alpha = \alpha(s)$ be a regular curve in $\mathbb{E}^4$ given with the arclength parameter s. If the position vector x lies in the hyperplane spanned by $\{T, N, W\}$ or equivalently the curvature function m_2 vanishes identically then α is called an osculating curve of first kind in $\mathbb{E}^4$.*

We obtain the following result.

Theorem 2.8 *Let* $\alpha = \alpha(s) : I \subset \mathbb{R} \to \mathbb{E}^4$ *be a unit speed regular curve in* $\mathbb{E}^4$ *given with incompressible canonical vector field* $\alpha(s)^{\mathrm{T}}$. *If* α *is an osculating curve of first kind then*

$$\kappa_1 = \pm \frac{1}{\sqrt{2as+c}}, \tag{30}$$

$$\frac{\kappa_2}{\kappa_3} = \mp \frac{b}{\sqrt{2as+c}}, \tag{31}$$

holds, where $m_0 = a$, $m_3 = b$ *and* c *are real constants.*

Proof. Let α be a regular curve with the incompressible canonical vector field $\alpha(s)^{\mathrm{T}}$. If α is an osculating curve of first kind then by definition $m_2 = 0$. So using the first equation of (29) we get the differential equation

$$\kappa_1' + \kappa_1^3 m_0 = 0,$$

which has a solution (30). Similarly using the second and third equation of (29) we get

$$m_3' = 0,$$

$$\frac{\kappa_2}{\kappa_3} = -m_3 \kappa_1,$$

where $m_0 = a, m_3 = b$ are constant functions. So we obtain (30).

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