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On Semigroup Ideals of Prime Near-Rings with Semiderivation

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Abstract

The notion of semiderivations of a ring was introduced by J. Bergen in [5]. Considerable work has been done on commutativity of prime near-rings with derivations in [2], [3] and [4]. In the present paper, it is shown that U is a nonzero semigroup ideal of $3-$prime near-ring N, d is a nonzero semiderivation associated with an additive mapping g of N such that $d(U)\subseteq Z$, then N is commutative ring. Also, we extend some well known results concerning semiderivations of prime rings for a semigroup ideal of prime near-rings.

Keywords: Prime Near Ring, Derivation, Semiderivation.

Yarıtürevli Asal Yakın Halkaların Yarıgrup İdealleri Üzerine

Özet

[5] te J. Bergen tarafından bir halkanın yarıtürevi tanımlanmıştır. [2], [3] ve [4] de türevli asal yakın halkaların komütatifliği ile ilgili bazı sonuçlar elde edilmiştir. Bu makalede, d g toplamsal dönüşümü ile belirlenmiş sıfırdan farklı bir yarıtürev olmak üzere N 3-asal yakın halkasının sıfırdan farklı bir U yarıgrup ideali için eğer $d(U)\subseteq Z$ ise bu durumda N nin değişmeli bir halka olduğu gösterilmiştir. Ayrıca

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yarıtürevli asal halkalarda bilinen bazı sonuçlar asal yakın halkaların yarıgrup idealleri için ispatlanmıştır.

Anahtar Kelimeler: Asal Yakın Halka, Türev, Yarıtürev.

1. Introduction

Throughout this paper, N will denote zero-symmetric left near-ring and Z its multiplicative center. Recall that a near-ring N is said to be $3-$prime if $xNy=(0)$ implies $x=0$ or $y=0$. For any $x,y\in N$, as usual $[x,y]=xy-yx$ will denote the well-known Lie product. A nonempty subset U of N will be called a semigroup right ideal (resp. semigroup left ideal) if $UN\subseteq U$ (resp. $NU\subseteq U$) and if U is both a semigroup right ideal and a semigroup left ideal, it will be called a semigroup ideal. As for terminologies used here without mention, we refer to G. Pilz [11].

Over the last seventeen years, many authors have proved commutativity theorems for prime or semiprime rings admitting derivations. In [5] J. Bergen has introduced the notion of semiderivation of a ring R which extends the notion of derivation of a ring R An additive mapping $d:R\to R$ is called a semiderivation if there exists a function $g:R\to R$ such that (i) $d(xy)=xd(y)+d(x)g(y)=g(x)d(y)+d(x)y$ and (ii) $d(g(x))=g(d(x))$ hold for all $x,y\in R.$ In case g is an identity map of $R,$ then all semiderivations associated with g are merely ordinary derivations. On the other hand, if g is a homomorphism of R such that $g\neq 1,$ then $d=g-1$ is a semiderivation which is not a derivation. In case R is prime and $d\neq 0,$ it has been shown by Chang [10] that g must necessarily be a ring endomorphism. Many authors studied commutativity on prime rings with semiderivation (see [8], [9] and [1] for a partial bibliography).

The study of derivations of near-rings was initiated by H. E. Bell and G. Mason in 1987 [2]. Some recent results on rings deal with commutativity on prime and semiprime rings admitting suitably-constrained derivations. Many authors have generalized the following identities: (i) $d(R)\subseteq Z,$ (ii) $d([x,y])=0,$ for all $x,y\in R$ where R is a ring or a near ring. In [6], A Boua et. al. have generalized these theorems for a semigroup ideal of $3-$prime near ring. We will extend these two results without considering g is as an auotomorphism. Also, we will prove some well known results for a semigroup ideal of prime near ring admitting semiderivation. The generalization is not trivial as the following example shows:

Example 1.1 *Let S be a $2-$torsion free left near ring and*

$$N=\left\{\begin{pmatrix}0 & x & y\\ 0 & 0 & 0\\ 0 & 0 & z\end{pmatrix} \mid x,y,z\in S\right\}.$$

Define maps $d,g:N\rightarrow N$ by

$$d\begin{pmatrix}0 & x & y\\ 0 & 0 & 0\\ 0 & 0 & z\end{pmatrix}=\begin{pmatrix}0 & 0 & y\\ 0 & 0 & 0\\ 0 & 0 & z\end{pmatrix},$$

$$g\begin{pmatrix}0 & x & y\\ 0 & 0 & 0\\ 0 & 0 & z\end{pmatrix}=\begin{pmatrix}0 & x & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

It can be verified that N be a left near ring and d is a semiderivation with associated a map g.

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2. Results

Lemma 2.1 [4, Lemma 1.3] *Let N be a $3-$prime near ring, U be a nonzero semigroup ideal of N and $x\in N$.*

i) If $Ux=(0)$ or $xU=(0)$, then $x=0$.

ii) If $[U,x]=(0)$, then $x\in Z$.

Lemma 2.2 [4, Lemma 1.4] *Let N be a $3-$prime near ring, U be a nonzero semigroup ideal of N and $a,b\in N$. If $aUb=(0)$, then $a=0$ or $b=0$.*

Lemma 2.3 [4, Lemma 1.5] *Let N be a $3-$prime near ring. If Z contains a nonzero semigroup ideal of N, then N is commutative ring.*

Lemma 2.4 [6, Lemma 2.3] *Let N be a near ring. If N has an additive mapping d, then the following conditions are equivalent:*

i) *d is a semiderivation associated with an additive mapping g,*

ii) $d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y)$ and $d(g(x)) = g(d(x))$ for all $x, y \in N$.

Lemma 2.5 [6, Lemma 2.4] *Let N be a prime near ring, d be a semiderivation associated with an additive mapping g of N. Then N satisfies the following partial distrubitive law:*

$$(xd(y) + d(x)g(y))g(z) = xd(y)g(z) + d(x)g(y)g(z), \quad \text{for all } x, y, z \in N.$$

The following Lemma is obtained from the above Lemma.

Lemma 2.6 *Let N be a prime near ring, d be a semiderivation associated with an automorphism g of N. Then N satisfies the following partial distrubitive law:*

$$(xd(y) + d(x)g(y))z = xd(y)z + d(x)g(y)z, \quad \text{for all } x, y, z \in N.$$

Lemma 2.7 [7, Theorem 1] *Let N be a $3-$prime near ring, U be a nonzero semigroup ideal of N, d be a semiderivation associated with an automorphism g of N. Then the following conditions are equivalent:*

i) $d(U) \subseteq Z$,

ii) *N is commutative ring.*

Lemma 2.8 *Let N be a $3-$prime near ring, U be a nonzero semigroup ideal of N and d be a semiderivation associated with an additive mapping g of N. If $d(U) = (0)$, then $d = 0$.*

Proof. Using Lemma 2.4, for any $u \in U, x \in N$, we get

$$0 = d(ux) = d(u)g(x) + ud(x),$$

and so

$$Ud(N) = (0).$$

By Lemma 2.1 (i), we have $d = 0$.

Lemma 2.9 *Let N be a $3-$prime near ring, U be a nonzero semigroup ideal of N, d be a nonzero semiderivation associated with an additive mapping g of N such that $d(U) \subseteq Z$. Then g is an homomorphism of N, that is*

$$g(xy) = g(x)g(y), \quad \text{for all } x, y \in N.$$

Proof. By the definition of d, we have

$$d(u(xy)) = ud(xy) + d(u)g(xy) \tag{1}$$

$$= uxd(y) + ud(x)g(y) + d(u)g(xy).$$

On the other hand, we get

$$d((ux)y) = uxd(y) + d(ux)g(y)$$

$$= uxd(y) + (ud(x) + d(u)g(x))g(y).$$

Applying Lemma 2.5, we arrive at

$$d((ux)y) = uxd(y) + ud(x)g(y) + d(u)g(x)g(y). \tag{2}$$

Comparing (1) and (2), we obtain that

$$d(u)g(xy) = d(u)g(x)g(y),$$

and so

$$d(u)(g(xy) - g(x)g(y)) = 0, \quad \text{for all } u \in U, x, y \in N.$$

Since $d(u) \in Z$, we find that

$$d(u) = 0 \text{ or } g(xy) - g(x)g(y) = 0, \quad \text{for all } u \in U, x, y \in N.$$

If $d(U) = (0)$, then $d = 0$ by Lemma 2.8. So, we must have

$$g(xy) = g(x)g(y), \quad \text{for all } x, y \in N.$$

Lemma 2.10 *Let N be $3-$a prime near ring, U be a nonzero semigroup ideal of N, d be a nonzero semiderivation associated with an additive mapping g of N such that $d(U) \subseteq Z$. Then N satisfies the following partial distrubitive law:*

$$(g(x)d(y) + d(x)y)z = g(x)d(y)z + d(x)yz, \quad \text{for all } x, y, z \in N.$$

Proof. Let $x, y, z \in N$, then by the definition of d we get

$$d(x(yz)) = g(x)d(yz) + d(x)yz$$

$$= g(x)g(y)d(z) + g(x)d(y)z + d(x)yz.$$

On the other hand, we calculate $d((xy)z)$ by using Lemma 2.9, we have

$$d((xy)z) = g(xy)d(z) + d(xy)z$$

$$= g(x)g(y)d(z) + d(xy)z.$$

Comparing the last two equations, we arrive at

$$d(xy)z = g(x)d(y)z + d(x)yz,$$

and so

$$(g(x)d(y) + d(x)y)z = g(x)d(y)z + d(x)yz, \quad \text{for all } x, y, z \in N.$$

Lemma 2.11 *Let N be a $3-$prime near ring, d be a nonzero semiderivation associated with an automorphism g of N. Then N satisfies the following partial distrubitive law:*

$$(g(x)d(y) + d(x)y)z = g(x)d(y)z + d(x)yz, \text{for all } x, y, z \in N.$$

Proof. Using the same arguments as in the proof of Lemma 2.10 and g is an automorphism of N, the partial distrubitive law follows.

The following theorem is generalization of [7, Theorem 1]. We prove this theorem without requiring that g is an auotomorphism.

Theorem 2.1 *Let N be a $3-$prime near ring, U be a nonzero semigroup ideal of N, d be a nonzero semiderivation associated with an additive mapping g of N. If $d(U) \subseteq Z$, then N is commutative ring.*

Proof. Commuting $d(uv)$ with $g(v)$, we have

$$(ud(v) + d(u)g(v))g(v) = g(v)(ud(v) + d(u)g(v)).$$

Using Lemma 2.5 and $d(u) \in Z$, we get

$$ud(v)g(v) + d(u)g(v)g(v) = g(v)ud(v) + d(u)g(v)g(v),$$

and so

$$ud(v)g(v) = g(v)ud(v), \quad \text{for all } u, v \in U.$$

By the hypothesis, we arrive at

$$d(v)[u, g(v)] = 0.$$

Since $d(v) \in Z$ and N is prime, we have for each $v \in U$,

$$d(v) = 0 \text{ or } [u, g(v)] = 0.$$

If $d(v) = 0$, then for any $v \in U$, $d(uv) = ud(v) + d(u)g(v)$, and so $d(u)g(v) \in Z$. Commuting this term with $y \in N$ and using $d(u) \in Z$, we obtain that

$$d(u)[g(v), y] = 0, \quad \text{for all } u \in U, y \in N.$$

Again using $d(u) \in Z$ and the primeness of N, we have $d(U) = (0)$ or $g(v) \in Z$. If $d(U) = (0)$, then by Lemma 2.8 we get $d = 0$, a contradiction. If $g(v) \in Z$, then we have $[u, g(v)] = 0$. Hence we arrive at $[u, g(v)] = 0$ for both cases. That is

$$[U, g(v)] = (0).$$

By Lemma 2.1 (ii), we obtain that $g(U) \subseteq Z$, and so $g(u)d(v) \in Z$.

Now, we commute $d(uv)$ with $y \in N$ and using Lemma 2.10, we get

$$(g(u)d(v) + d(u)v)y = y(g(u)d(v) + d(u)v),$$

$$g(u)d(v)y + d(u)vy = yg(u)d(v) + yd(u)v.$$

Since $g(u)d(v), d(u) \in Z$, we arrive at

$$d(u)[v, y] = 0, \quad \text{for all } u, v \in U, y \in N,$$

and so

$$d(U) = (0) \text{ or } [U, N] = (0).$$

If $d(U) = (0)$, then by Lemma 2.8, we have $d = 0$, a contradiction. If $[U, N] = (0)$, then $N \subseteq Z$ by Lemma 2.1 (ii), and so N is commutative ring by Lemma 2.3.

Lemma 2.12 *Let N be a $3-$prime near ring, U be a nonzero semigroup ideal of N, d be a semiderivation associated with an additive mapping g of N and $a \in N$. If $ad(U) = (0)$, then $a = 0$ or $d = 0$.*

Proof. By the hypothesis and Lemma 2.4, for any $u \in U, x \in N$, we get

$$0 = ad(ux) = ad(u)g(x) + aud(x).$$

Using the hypothesis, we have

$$aUd(x) = (0), \quad \text{for all } x \in N.$$

By Lemma 2.2, we find that $a = 0$ or $d = 0$.

Lemma 2.13 *Let N be a $3-$prime near ring, U be a nonzero semigroup ideal of N, d be a semiderivation associated with an automorphism g of N and $a \in N$. If $d(U)a = (0)$, then $a = 0$ or $d = 0$.*

Proof. For any $u \in U, x \in N$, we get

$$0 = d(xu)a = (xd(u) + d(x)g(u))a.$$

Using Lemma 2.6 and the hypothesis, we have

$$0 = xd(u)a + d(x)g(u)a,$$

and so

$$d(x)g(U)a = (0).$$

We can write the last equation such as

$$d(x)Ia = (0),$$

where $I = g(U)$. By Lemma 2.2, we find that $a = 0$ or $d = 0$ or $I = g(U) = (0)$. If $g(U) = (0)$, then $U = (0)$, a contradiction. So, we must have $a = 0$ or $d = 0$.

Theorem 2.2 *Let N be a $3-$prime near ring, U be a nonzero semigroup ideal of N and d be a semiderivation associated with an additive mapping g of N. If $[d(u), v] \in Z$, for all $u, v \in U$, then N is commutative ring.*

Proof. Replacing v by $d(u)v$ in the hypothesis, we have

$$[d(u), d(u)v] \in Z.$$

That is

$$d(u)[d(u), v] \in Z, \quad \text{for all } u, v \in U.$$

Commuting this term with $v \in U$ and using $[d(u), v] \in Z$, we get $[d(u), v]^2 = 0$. Again using $[d(u), v] \in Z$, we conclude that $[d(u), v] = 0$, for all $u, v \in U$. Thus we get $d(U) \subseteq Z$ by Lemma 2.1 (ii), and so N is commutative ring from Theorem 2.1.

Theorem 2.3 *Let N be a $3-$prime near ring, U be a nonzero semigroup ideal of N and d be a semiderivation associated with an automorphism g of N. If d acts as a homomorphism on U, then $d = 0$.*

Proof. Let d acts as a homomorphism on U. Then

$$d(uv) = g(u)d(v) + d(u)v = d(u)d(v), \quad \text{for all } u, v \in U.$$

Replacing v by vw in this equation, we get

$$g(u)d(vw) + d(u)vw = d(u)d(vw)$$
$$= d(u)d(v)d(w)$$
$$= d(uv)d(w)$$
$$= (g(u)d(v) + d(u)v)d(w).$$

Applying Lemma 2.11 in the right of the last equation, we have

$$g(u)d(vw) + d(u)vw = g(u)d(v)d(w) + d(u)vd(w)$$
$$= g(u)d(vw) + d(u)vd(w)$$

and so

$$d(u)U(w - d(w)) = (0), \quad \text{for all } u, w \in U.$$

By Lemma 2.2, we have either $d(U) = (0)$ or $w = d(w),$ for all $w \in U.$ If $d(U) = (0),$ then $d = 0$ by Lemma 2.8.

Suppose $d(w) = w,$ for all $w \in U.$ Hence by Lemma 2.4, we get

$$uv = d(uv) = d(u)v + g(u)d(v)$$
$$= uv + g(u)v$$

and so

$$g(U)U = (0).$$

Applying Lemma 2.1 (i), we have $g(U) = (0).$ Since g is an automorphism of $N,$ we find that $U = (0),$ a contradiction. So we obtain that $d = 0.$

Theorem 2.4 *Let N be a $3-$prime near ring, U be a nonzero semigroup ideal of N and d be a semiderivation associated with an automorphism g of N. If d acts as an anti-homomorphism on $U,$ then $d = 0$.*

Proof. By the hypothesis, we get

$$d(uv) = ud(v) + d(u)g(v) = d(v)d(u), \quad \text{for all } u, v \in U.$$

Replacing v by uv in the last equation, then

$$ud(uv)+d(u)g(uv)=d(uv)d(u)$$

$$=(ud(v)+d(u)g(v))d(u).$$

Using Lemma 2.6 the right of the last equation, we have

$$ud(uv)+d(u)g(uv)=ud(v)d(u)+d(u)g(v)d(u).$$

Since d is as an anti-homomorphism on U, we get

$$ud(uv)+d(u)g(uv)=ud(uv)+d(u)g(v)d(u)$$

and so

$$d(u)g(u)g(v)=d(u)g(v)d(u), \quad \text{for all } u,v\in U.$$

Since g is an automorphism of N, this equation shows that

$$d(u)g(u)j=d(u)jd(u), \quad \text{for all } u\in U,\ j\in I,$$

where $I=g(U)$. It is clear that I is a semigroup ideal of N. Writing $jx, x\in N$ instead of j in the last equation and using this, we have

$$d(u)j[d(u),x]=0, \quad \text{for all } u\in U,\ j\in I, x\in N.$$

By Lemma 2.2, this implies that $d(u)=0$ or $[d(u),x]=0$, and so $d(U)\subseteq Z$. Thus d acts as a homomorphism on U, and so $d=0$ by Theorem 2.3.

Theorem 2.5 *Let N be a $3-$prime near ring, U be a nonzero semigroup ideal of N and d be a semiderivation associated with an automorphism g of N. If $d([u,v])=[d(u),v]$, for all $u,v\in U$, then N is commutative ring.*

Proof. By the hypothesis, we have

$$d(uv-vu)=d(u)v-vd(u),$$

$$d(u)v+g(u)d(v)-(vd(u)+d(v)g(u))=d(u)v-vd(u),$$

$$g(u)d(v)-d(v)g(u)-vd(u)=-vd(u)$$

and so

$$[g(u),d(v)]=0, \quad \text{for all } u,v\in U. \tag{3}$$

Since g is an automorphism of N, this equation shows that

$$[I,d(v)]=(0), \quad \text{for all } v\in U,$$

where $I = g(U)$. It is clear that I is a semigroup ideal of N. Using Lemma 2.1 (ii), we get $I = g(U) = (0)$ or $d(U) \subseteq Z$. If $g(U) = (0)$, then $U = (0)$, a contradiction. If $d(U) \subseteq Z$, then N is commutative ring by Lemma 2.7.

Theorem 2.6 *Let N be a $3-$prime near ring, U be a nonzero semigroup ideal of N and d be a semiderivation associated with an automorphism g of N. If $d([u,v]) = [u, d(v)]$, for all $u, v \in U$, then N is commutative ring.*

Proof. Expanding our hypothesis, we get

$$d(uv - vu) = ud(v) - d(v)u,$$

$$ud(v) + d(u)g(v) - (d(v)u + g(v)d(u)) = ud(v) - d(v)u,$$

$$d(u)g(v) - g(v)d(u) - d(v)u = -d(v)u$$

and so

$$[d(u), g(v)] = 0, \quad \text{for all } u, v \in U.$$

Now applying the same arguments as used after equation (3) in the proof of Theorem 2.5, we get the required result.

Theorem 2.7 *Let N be a $3-$prime $2-$torsion free near ring, U be a nonzero semigroup ideal of N, d be a semiderivation associated with an automorphism g of N. If $d^2(U) = (0)$, then $d = 0$.*

Proof. For arbitrary $u, v \in U$, we have

$$0 = d^2(uv) = d(d(uv)) = d(ud(v) + d(u)g(v))$$

$$= ud^2(v) + d(u)g(d(v)) + d^2(u)g^2(v) + d(u)d(g(v)).$$

By the hypothesis,

$$d(u)g(d(v)) + d(u)d(g(v)) = 0, \quad \text{for all } u, v \in U.$$

Using $dg = gd$, we get

$$2d(u)g(d(v)) = 0, \quad \text{for all } u, v \in U.$$

Since N is a $2-$torsion free near ring, we have

$$d(u)g(d(v)) = 0, \quad \text{for all } u, v \in U.$$

By Lemma 2.13, we obtain that $d(U)=(0)$ or $g(d(U))=(0)$, and so $d(U)=(0)$. Hence we get $d=0$ by Lemma 2.8.

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