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# On Δ-Uniform and Δ-Pointwise Convergence on Time Scale

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**Abstract**

In this article, we define the concept of Δ-Cauchy, Δ-uniform convergence and Δ-pointwise convergence of a family of functions $\{f_j\}_{j\in\mathbb{J}}$, where $\mathbb{J}$ is a time scale. We study the relationships between these notions. Moreover, we introduced sufficient conditions for interchangeability Δ-limitation with Riemann Δ-integration or Δ-differentiation. Also, we obtain the analogue of the well-known Dini's Theorem.

***Keywords:*** Δ-Convergence; Δ-Cauchy; Statistical convergence.

## Zaman Skalası Üzerinde Δ-Düzgün ve Δ-Noktasal Yakınsaklık

**Öz**

Bu makalede $\mathbb{J}$ bir zaman skalası olmak üzere, $\{f_j\}_{j\in\mathbb{J}}$ fonksiyon ailesi için Δ-Cauchy, Δ-düzgün yakınsaklık ve Δ-noktasal yakınsaklık kavramları verilerek bu kavramlar arasındaki ilişkiler incelenmiştir. Δ-limit ile Riemann Δ-integrali ve Δ-türevin yer değişme problemi araştırılarak Dini Teoreminin farklı bir versiyonu elde edilmiştir.

***Anahtar Kelimeler:*** Δ-Yakınsaklık; Δ-Cauchy; İstatistiksel yakınsaklık.

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## 1. Introduction and Preliminaries

The time scale calculus was introduced in 1989 by German mathematician Stefan Hilger [1]. It is a unification of the theory of differential equations with that of difference equations. This theory was developed to a certain extent in [2] by Hilger.

The notion of statistical convergence for complex number sequences was introduced by Fast in [3]. Schoenberg gave some properties of this concept [4]. Fridy progressed with the statistically Cauchy and showed the equivalence of these concepts in [5].

In recent years, there are many studies based on the density function, which is defined on some subsets of time scale. For instance, first author and Tan [6] gave the notions of $\Delta$-Cauchy and $\Delta$-convergence of a function defined on time scale by using $\Delta$-density. The notion of $m$- and $(\lambda, m)$- uniform density of a set and the concept of $m$- and $(\lambda, m)$- uniform convergence on a time scale were presented by Altin et al. [7]. Also, Altin et al. gave $\lambda$-statistical convergence on time scale and examined some of its features [8]. Some fundamental properties of Lacunary statistical convergence and statistical convergence on time scale investigated by Turan and Duman in [9].

Let $\mathcal{S}$ be the collection of all subsets of time scale $\mathbb{J}$ in the form of $[a, b)$, where $[a, a) = \emptyset$. Then $\mathcal{S}$ is a semiring on $\mathbb{J}$. The set function $m$ defined by $m([a, b)) = b - a$ is a measure on $\mathcal{S}$. The outer measure $m^*: \mathcal{S} \to [0, \infty]$ generated by $m$ is defined by

$$m^*(A) := \inf\left\{\sum_{n=1}^{\infty} (b_n - a_n): A \subset \bigcup_{n=1}^{\infty} [a_n, b_n)\right\}.$$

The family of all $m^*$-measurable (it is also called $\Delta$-measurable) sets $\mathcal{M} = \mathcal{M}(m^*)$ is a $\sigma$-algebra and it is well known that from the measure theory the restriction of $m^*$ to $\mathcal{M}$, which we denote by $\mu_\Delta$, is a measure. This measure is called Lebesgue $\Delta$-measure on $\mathbb{J}$.

**Definition 1.** [5] Let $A \subset \mathbb{N}$, and

$$A_n = \sum_{m \leq n, m \in A} 1.$$

The asymptotic density of $A$ is defined by $\delta(A) = lim_n n^{-1} A_n$, which is also called natural density. The real number sequence $x = (x_n)$ is statistically convergent to $l$ if for each $\epsilon > 0$, $\delta(\{n \in \mathbb{N}: |x_n - l| \geq \epsilon\} = 0$; in this case we write st-$lim\ x = l$.

From now on we assume that *sup* $\mathbb{J} = \infty$ and $\mathbb{J}$ has a minimum for the time scale $\mathbb{J}$.

**Definition 2.** ($\Delta$ -Density) [6] Let $B$ be a subset of $\mathbb{J}$ such that $B \in \mathcal{M}$ and $a = \min \mathbb{J}$. $\Delta$-density of $B$ in $\mathbb{J}$ is defined by

$$\delta_{\Delta}(B) := \lim_{j \to \infty} \frac{\mu_{\Delta}(B \cap [a, j])}{\sigma(j) - a}$$

provided that this limit exists.

A property of points of $\mathbb{J}$ is said to hold $\Delta$-almost everywhere (or $\Delta$-almost all $j \in \mathbb{J}$) if the set of points in $\mathbb{J}$ at which it fails to hold has zero $\Delta$-density. The expression $\Delta$-almost everywhere abbreviated to $\Delta$-a.e.

**Definition 3.** ($\Delta$ -Convergence) [6] If for every $\epsilon > 0$, the inequality $|g(j) - l| < \epsilon$ holds $\Delta$-a.e. on $\mathbb{J}$, then $g: \mathbb{J} \to \mathbb{R}$ is called $\Delta$-convergent to $l \in \mathbb{R}$ (or has $\Delta$-limit). In this case we write $\Delta\text{-}lim_{j \to \infty} f(j) = l$.

**Definition 4.** ($\Delta$-Cauchy) [6] The function $g: \mathbb{J} \to \mathbb{R}$ is $\Delta$-Cauchy provided that for each $\epsilon > 0$, there exist $K = K(\epsilon) \subset \mathbb{J}$ and $j_0 \in \mathbb{J}$ such that $\delta_{\Delta}(K) = 1$ and $|g(j) - g(j_0)| < \epsilon$ holds for all $j \in K$.

Note that the $\Delta$-density, $\Delta$-Cauchy and $\Delta$-Convergence coincide with the natural density, statistical Cauchy and statistical convergence respectively whenever $\mathbb{J}$ is the natural numbers.

## 2. Δ-Pointwise and Δ-Uniform Convergence

In this section, we will deal with the family of functions $\{f_j\}_{j \in \mathbb{J}}$ whose elements defined on any subset of real numbers.

**Definition 5.** ($\Delta$-Pointwise Convergence) Let $B \subset \mathbb{R}$ and for each $j \in \mathbb{J}$, $f_j$ and $f$ be real valued functions on $B$. The family $\{f_j\}_{j \in \mathbb{J}}$ converges $\Delta$-pointwise to $f$ on B, if for each given $\epsilon > 0$ and $t \in B$, the inequality $|f_j(t) - f(t)| < \epsilon$ holds $\Delta$-a.e. on $\mathbb{J}$. This notion is abbreviated as $\{f_j\}_{j \in \mathbb{J}} \to f$ on $B$.

**Definition 6.** ($\Delta$-Uniform Convergence) Let $B \subset \mathbb{R}$ and for each $j \in \mathbb{J}$, $f_j$ and $f$ be real valued functions on $B$. The family $\{f_j\}_{j \in \mathbb{J}}$ converges $\Delta$-uniformly to $f$ on $B$, if for each given $\epsilon > 0$, the inequality $|f_j(t) - f(t)| < \epsilon$ holds $\Delta$-a.e. on $\mathbb{J}$ and for all $t \in B$. In this case we write $\{f_j\}_{j \in \mathbb{J}} \rightrightarrows f$ on $B$.

**Definition 7.** (Δ-Uniform Cauchy) Let $B \subset \mathbb{R}$ and $\{f_j\}$ be a family of real valued functions defined on $B$. The family $\{f_j\}_{j\in\mathbb{J}}$, Δ-uniform Cauchy on $B$, if for all $\epsilon > 0$ there exists a subset $K = K(\epsilon)$ of $\mathbb{J}$ and $j_0 \in \mathbb{J}$ such that $\delta_\Delta(K) = 1$ and $|f_j(t) - f_{j_0}(t)| < \epsilon$ for all $j \in$ K and for all $t \in B$.

**Example 8.** Let $\mathbb{J} = [0,\infty)$ and $B \subset \mathbb{R}$. We denote the irrational and rational numbers in $[0,\infty)$ by $\mathbb{I}_{[0,\infty)}$ and $\mathbb{Q}_{[0,\infty)}$, respectively. We consider the functions $f_j: B \to \mathbb{R}$ $(j \in \mathbb{J})$ defined as;

$$f_j(t) = \begin{cases} \sin jt, & j \in \mathbb{Q}_{[0,\infty)} \\ 0, & j \in \mathbb{I}_{[0,\infty)} \end{cases}.$$

Since the set $\mathbb{Q}_{[0,\infty)}$ has zero density in $\mathbb{J}$, the density of $\mathbb{I}_{[0,\infty)}$ is one. Hence, $\{f_j\}_{j\in\mathbb{J}} \rightrightarrows f = 0$ on $B$.

It is easily seen that Δ-uniform convergence implies Δ-pointwise convergence, but the converse is not always true as we can see from the following counter-examle.

**Example 9.** Let $\mathbb{J} = [1,\infty)$ and $j \in \mathbb{J}$. Consider the functions $f_j: [0,\infty) \to \mathbb{R}$ defined as;

$$f_j(t) = \begin{cases} \frac{t}{j}, & j \in \mathbb{Q}_{[1,\infty)} \\ 0, & j \in \mathbb{I}_{[0,\infty)} \end{cases}.$$

Although $\{f_j\}_{j\in\mathbb{J}}$ is Δ-pointwise convergent to $f = 0$, it is not Δ-uniform convergent.

The proof of the following theorem is clear.

**Theorem 10.** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real valued functions defined on $B \subset \mathbb{R}$. If $(f_n)_{n\in\mathbb{N}}$ converges uniformly (pointwise) to $f$, then $\{f_n\}_{n\in\mathbb{N}}$ converges Δ-uniformly (Δ-pointwise) to $f$.

**Theorem 11.** Let $\{f_j\}_{j\in\mathbb{J}}$ be a family of real valued functions defined on $B \subset \mathbb{R}$. If $\{f_j\}_{j\in\mathbb{J}} \to f$ on $B$, then $\{f_j\}_{j\in\mathbb{J}} \rightrightarrows f$ on $B$ if and only if

$$\Delta - \lim_{j\to\infty} \sup_{t\in B} |f_j(t) - f(t)| = 0.$$

**Theorem 12.** Let $\{f_j\}_{j\in\mathbb{J}}$ be a family of real valued functions defined on $B \subset \mathbb{R}$. $\{f_j\}_{j\in\mathbb{J}} \rightrightarrows f$ on $B$ if and only if it is Δ-uniform Cauchy on $B$.

**Proof.** Necessity is obvious. Let $\{f_j\}_{j\in\mathbb{J}}$ be Δ-uniform Cauchy on $B$. For a given $\epsilon > 0$ there exists $j_0 \in \mathbb{J}$ and $K \subset \mathbb{J}$ such that $\delta_\Delta(K) = 1$, the inequality

$$|f_j(t) - f_{j_0}(t)| < \frac{\epsilon}{2}, \tag{1}$$

holds for all $j \in K$ and $t \in B$. Let $g_t\colon \mathbb{J} \to \mathbb{R}$ defined by $g_t(j) = f_j(t)$ for each $t \in B$. For each fixed $t$

$$|g_t(j) - g_t(j_0)| = |f_j(t) - f_{j_0}(t)| < \epsilon,$$

holds Δ-a.e. on $\mathbb{J}$. Therefore, the functions $g_t$, $(t \in B)$ are Δ-Cauchy. These functions have Δ-limit. Let $f(t) = \Delta\text{-}\lim_{j\to\infty} g_t(j)$. As $j \to \infty$, the Δ-limit of (1) yields

$$|f(t) - f_{j_0}(t)| \leq \frac{\epsilon}{2}. \tag{2}$$

In view of inequalities (1) and (2), one can get

$$\left|f_j(t) - f(t)\right| \leq \left|f_j(t) - f_{j_0}(t)\right| + \left|f_{j_0}(t) - f(t)\right| < \epsilon,$$

for all $j \in K$ and for all $t \in B$.

**Theorem 13.** Let $\mathbb{T}$ and $\mathbb{J}$ be two time scales and $[\alpha, \beta] \subset B \subset \mathbb{T}$. If $f_j \in C_{rd}(B, \mathbb{R}) := \{f | f\colon B \to \mathbb{R}\ is\ rd - continuous\}$ for all $j \in \mathbb{J}$, and $\{f_j\}_{j\in\mathbb{J}} \rightrightarrows f$, then $f \in C_{rd}(B, \mathbb{R})$ and

$$\Delta - \lim_{j\to\infty} \int_\alpha^\beta f_j(t)\Delta t = \int_\alpha^\beta f(t)\Delta t.$$

**Proof.** Let any positive $\epsilon$ be given. In accordance with Δ-uniform convergence, the time scale $\mathbb{J}$ has a subset $K$ such that $\delta_\Delta(K) = 1$ and the inequality

$$|f_j(t) - f(t)| < \frac{\epsilon}{3},$$

holds for all $j \in K$ and for all $t \in B$.

Let $j_0 \in K$ and $t_0 \in B$ are arbitrary. We consider two cases. In the first case we assume that $t_0$ is left-dense. From rd-continuity of $f_{j_0}$, we can find $\delta > 0$ such that

$$|f_{j_0}(\xi) - f_{j_0}(\eta)| < \frac{\epsilon}{3},$$

for any $\xi, \eta \in (t_0 - \delta, t_0)$. If $t_n \to t_0^-$ as $n \to \infty$, then there exists natural number $n_0$ such that $n, m > n_0$ imply $t_m, t_n \in (t_0 - \delta, t_0)$ and

$$|f_{j_0}(t_n) - f_{j_0}(t_m)| < \frac{\epsilon}{3}. \tag{3}$$

Hence, for $m, n > n_0$, we have

$$\begin{aligned} |f(t_n) - f(t_m)| &= |f(t_n) - f_{j_0}(t_n) + f_{j_0}(t_n) - f_{j_0}(t_m) + f_{j_0}(t_m) - f(t_m)| \\ &\leq |f(t_n) - f_{j_0}(t_n)| + |f_{j_0}(t_n) - f_{j_0}(t_m)| \\ &\quad + |f_{j_0}(t_m) - f(t_m)| \\ &< \epsilon. \end{aligned} \tag{4}$$

Therefore, the function $f$ has finite left-sided limit at $t_0$.

In the second case we assume that $t_0$ is right-dense. Then all functions $f_j$ are continuous at $t_0$. If $t_n \to t_0$ as $n \to \infty$, then there exists natural number $n_0$ such that $n, m > n_0$ imply $t_m, t_n \in (t_0 - \delta, t_0 + \delta)$ and (3-4) holds. This is implies continuity of $f$ at $t_0$. Therefore, $f$ is Riemann $\Delta$-integrable on every subinterval $[\alpha, \beta] \subset B$. So, we obtain the inequality

$$\left| \int_\alpha^\beta f_j(t)\Delta t - \int_\alpha^\beta f(t)\Delta t \right| \leq \int_\alpha^\beta |f_j(t) - f(t)| \Delta t < \frac{\epsilon}{3}(\beta - \alpha),$$

for every $j \in K$ that completes our proof.

**Theorem 14.** Let $\mathbb{T}$ and $\mathbb{J}$ be two time scales and $[\alpha, \beta] \subset \mathbb{T}$. Suppose that the functions

$$f_j : [\alpha, \beta] \to \mathbb{R} \qquad (j \in \mathbb{J})$$

satisfies the following conditions on $[\alpha, \beta]$ :

1. $f_j$ has Hilger derivative and its Hilger derivative $f_j^\Delta$ is rd-continuous,
2. $\{f_j\}_{j \in \mathbb{J}} \to f$,
3. $\{f_j^\Delta\}_{j \in \mathbb{J}} \rightrightarrows g$.

Then $f$ has Hilger derivative on $[\alpha, \beta]$ and $f^\Delta(t) = g(t)$ for all $t \in [\alpha, \beta]$.

**Proof.** $g$ is rd-continuous on $[\alpha, \beta]$ by Theorem 13 and so $g$ is Riemann $\Delta$-integrable on this interval. By the help of Theorem 13, we have

$$\int_\alpha^t g(s)\Delta s = \Delta - \lim_{j\to\infty} \int_\alpha^t f_j^\Delta f(s)\Delta s = \Delta - \lim_{j\to\infty} (f_j(t) - f_j(\alpha)) = f(t) - f(\alpha),$$

for all $t \in [\alpha, \beta]$. Since the left hand-side of the last equality has Hilger derivative, the right hand-side also has, and it follows that $f^{\Delta}(t) = g(t)$ for all $t \in [\alpha, \beta]$.

**Theorem 15.** (Dini's Theorem**)** *Let* $X$ be a compact metric space. Let $f: X \to \mathbb{R}$ be a continuous function and the functions $f_j: X \to \mathbb{R}$, $(j \in \mathbb{J})$ are continuous for $\Delta$-almost all $\mathbb{J}$. If the following two conditions are satisfied:

1. $\{f_j\}_{j \in \mathbb{J}} \to f$ on $X$,

2. $f_j(x) \leq f_i(x)$ for all $x \in X$ and $\Delta$-almost all $i, j \in \mathbb{J}$ such that $i < j$,

then $\{f_j\}_{j \in \mathbb{J}} \rightrightarrows f$ on $X$.

**Proof.** There exists a subset $K_1 \subset \mathbb{J}$ with $\Delta$-density 1. Moreover, for each $j \in K_1$ the functions $f_j$ are continuous, and

$$f_j(x) \leq f_i(x) \quad \text{for all} \quad x \in X,$$

holds for all $i, j \in K_1$ such that $i < j$. For each $j \in K_1$, define $g_j = f_j - f$. Then $\{g_j\}_{j \in K_1}$ is a family of continuous functions on the compact metric space $X$ that converges $\Delta$-pointwise to 0. Furthermore,

$$0 \leq g_j(x) \leq g_i(x),$$

for all $x \in X$ and $i, j \in K_1$ such that $i < j$.

Let $\epsilon > 0$ and define

$$G_j = \{x \in X : g_j(x) < \epsilon\}, \quad (j \in \ K_1).$$

Since $g_j$ is continuous, then $G_j$ is an open set and $G_i \subset G_j$ for each $i, j \in K_1$ such that $i < j$.

Let $x_0 \in X$ be arbitrary. Since $\Delta\text{-}\lim_{j \to \infty} g_j(x_0) = 0$, then there exists a subset $K_2 \subset \mathbb{J}$ such that $\delta_{\Delta}(K_2) = 1$ and the inequality $|g_j(x_0)| < \epsilon$ holds for all $j \in K_2$. If we set $K = K_1 \cap K_2$ then $\delta_{\Delta}(K) = 1$ and $g_j(x_0) = |g_j(x_0)| < \epsilon$ for all $j \in K$. Thus $x_0 \in G_j$ for all $j \in K$, and thus, we have

$$X = \bigcup_{j \in K} G_j.$$

Since $K$ is compact and $G_i \subset G_j$ when $i < j$, then there is a $j_0 \in K$ with $G_{j_0} = X$. Then we have $G_j = X$ for all $j \in K$ such that $j > j_0$. This implies that $f_j(x) - f(x) = g_j(x) < \epsilon$ for all $x \in X$ and $j \in K$ such that $j > j_0$. Consequently, $\{f_j\}_{j\in\mathbb{J}} \rightrightarrows f$ on $X$.

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