Adıyaman University Journal of Science
https://dergipark.org.tr/en/pub/adyujsci
DergiPark AKADEMİK
ISSN 2147-1630
e-ISSN 2146-586X

Inverse Spectral Problems for Second Order Difference Equations with Generalized Function Potentials by aid of Parseval Formula

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Received: 19.07.2019 **Accepted:** 03.03.2020 **Published:** 25.06.2020

Abstract

In the present study we are investigated inverse spectral problems for spectral analysis and two spectra of matrix J by using equality which is equivalance Parseval formula. The matrix J is $N \times N$ tridiagonal almost-symmetric matrix. The mean of almost-symmetric is the entries above and below the main diagonal are the same except the entries a_M and c_M.

Keywords: Parseval formula; Spectral analysis; Two spectra.

Parseval Formülü yardımıyla Genelleşmiş Fonksiyon Katsayılı İkinci Mertebeden Fark Denklemleri için Ters Spektral Problemler

Öz

Bu çalışmada, Parseval formülü ile eşdeğer olan eşitlik kullanılarak J matrisinin spektral analizine göre ve iki spektrumuna göre ters spektral problemleri incelenmiştir. J, $N \times N$ tipinde hemen hemen simetrik üçköşegensel matristir. Hemen hemen simetriklik, a_M ve c_M elemanları dışında matrisin köşegeninin altında ve üstündeki elemanları eşit olmasıdır.

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DOI: 10.37094/adyujsci.594199

Anahtar Kelimeler: Parseval formülü; Spektral analiz; İki spektrum.

1. Introduction

Consider the second order difference equation

$$a_{n-1}y_{n-1}+b_n y_n+a_n y_{n+1}=\lambda\rho_n y_n,\quad a_{-1}=c_{N-1}=1,\ n\in\{0,1,...,N-1\}, \tag{1}$$

with the boundary conditions

$$y_{-1}=y_N=0, \tag{2}$$

where $y=\{y_n\}_{n=0}^{N-1}$ is column vector which is solution of the second order difference equation, ρ_n is constant

$$\rho_n=\begin{cases}1, & 0\le n\le M\\ \alpha, & M<n\le N-1\end{cases},\ \alpha\in\mathbb{R}^+-\{1\}, \tag{3}$$

and $a_n,\ b_n\in\mathbb{R},\quad a_n>0$ are coefficients of Eqn. (1),

$$\begin{aligned}&c_n=a_n/\alpha,\quad n\in\{M,M+1,...,N-2\},\\ &d_n=b_n/\alpha,\quad n\in\{M+1,M+2,...,N-1\}.\end{aligned} \tag{4}$$

Now, we can write the Eqn. (1) by definition of ρ_n

$$\begin{cases}a_{n-1}y_{n-1}+b_n y_n+a_n y_{n+1}=\lambda y_n, & n\in\{0,1,...,M\},\\ c_{n-1}y_{n-1}+d_n y_n+c_n y_{n+1}=\lambda y_n, & n\in\{M+1,M+2,...,N-1\}.\end{cases}$$

J is $N\times N$ tridiagonal almost-symmetric matrix and the entries of J are the coefficients of Eqn. (1).

$$J=\begin{bmatrix} b_0 & a_0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{M-1} & a_{M-1} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & a_{M-1} & b_M & a_M & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & c_M & d_{M+1} & c_{M+1} & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & c_{M+1} & d_{M+2} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & d_{N-3} & c_{N-3} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & c_{N-3} & d_{N-2} & c_{N-2} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & c_{N-2} & d_{N-1} \end{bmatrix}.$$

So, the eigenvalue problem $Jy = \lambda y$ is equivalent problem (1)-(3) which is discrete form Sturm-Liouville problem with discontinuous coefficients

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}y(x)\right]+q(x)y(x)=\lambda\rho(x)y(x), \quad x\in[a,b], \qquad (5)$$
$$y(a)=y(b)=0,$$

where $\rho(x)$ is a piecewise function

$$\rho(x)=\begin{cases}1, & a\le x\le c \\ \alpha^2, & \mathrm{c}<x\le b\end{cases}, \qquad \alpha^2\ne 1.$$

H. Hochstadt made significant contributions to the development of the inverse problem for second order difference equations. He studied inverse problem for Jacobi matrices in [1-4]. Later, G. Guseinov has pioneered for inverse problem of infinite symmetric tridiagonal matrices. He considered different kinds of inverse spectral problems for second order difference equation; such as the inverse spectral problems of spectral analysis for infinite Jacobi matrices in [5], the inverse spectral problems for the infinite non-self adjoint Jacobi matrices from generalized spectral function in [6, 7], and the inverse spectral problems for same matrices from spectral data and two spectra in [8-12]. The inverse spectral problem for discrete form of Sturm-Liouville problem with continuous coefficients has been studied in [13] and the inverse spectral problem with spectral parameter in the initial conditions has been studied by M. Manafov in [14]. The eigenvalues and eigenfunctions and the inverse problem for Sturm-Liouville operator with discontinuous coefficients which is the same problem given by (5) are investigated by E. Akhmedova and H.

Huseynov in [15, 16], respectively. Bala et al. are studied inverse spectral problem for almost symmetric tridiagonal matrices from generalized spectral function in [17] and they examined inverse spectral problems for same matrices from spectral data and two spectra in [18]. Finite dimensional inverse problems are investigated by H. Huseynov in [19].

Also, Parseval equality of discrete Sturm-Liouville equation with periodic generalized function potentials is studied by Manafov et al. in [20]. At the same time a new approach for higher-order difference equations and eigenvalue problems is examined by Bas and Ozarslan in [21].

The goal of this article is to study inverse spectral problems of the problem (1)-(3) for spectral analysis and two spectra by using Parseval formula.

2. Direct Problem for Spectral Analysis

The matrix J has N number eigenvalues $\lambda_1, \lambda_2, ..., \lambda_N$ and N number eigenvectors $v_1, v_2, ..., v_N$, which form an orthonormalized basis. Assume that the eigenvalues are real. We bring to mind the algorithm of structure for the matrix J eigenvalues and eigenvectors.

Let $P_n(\lambda)$ be a solution of Eqn. (1)

$$a_{n-1}P_{n-1}(\lambda) + b_n P_n(\lambda) + a_n P_{n+1}(\lambda) = \lambda \rho_n P_n(\lambda), \quad n \in \{0,1,...,N-1\}, \tag{6}$$

with initial conditions

$$P_{-1}(\lambda) = 0, \;\; P_0(\lambda) = 1, \tag{7}$$

and the degree of polynomial $P_n(\lambda)$ is n.

Lemma 1. The following equality holds:

$$\det(J - \lambda I) = (-1)^N a_0 a_1 ... a_M c_{M+1} ... c_{N-1} P_N(\lambda).$$

Proof. See [17].

According to Lemma 1, the roots of the equation $P_n(\lambda)$ are equal the eigenvalues of J, and eigenvectors corresponding eigenvalues λ_k, $k = \overline{1,N}$ will be

$$\Re\left(\lambda_k\right)=\left(P_0\left(\lambda_k\right),P_1\left(\lambda_k\right),...,P_{N-1}\left(\lambda_k\right)\right)^T.$$

Assuming that $v_k=\dfrac{\Re\left(\lambda_k\right)}{\sqrt{\beta_k}}$, where $\beta_k=\sum_{j=0}^{N-1}P_j^2\left(\lambda_k\right)$. Thus, we have the complete orthonormalized system of eigenvectors of the matrix J. The numbers β_k are called normalized numbers of the problem (1)-(3).

Lemma 2. Eigenvalues of matrix J are different.

Proof. Because of eigenvalues λ_k, $k=1,2,...,N$ are the roots of polynomial $P_N\left(\lambda\right)$, we must show that $P_N'\left(\lambda_k\right)\neq 0$. Firstly, take the derivative equation Eqn. (6) by λ, we have

$$a_{n-1}P_{n-1}'\left(\lambda\right)+b_nP_n'\left(\lambda\right)+a_nP_{n+1}'\left(\lambda\right)=\lambda\rho_nP_n'\left(\lambda\right)+\rho_nP_n\left(\lambda\right). \tag{8}$$

Now, if the Eqn. (8) is multiplied by $P_n\left(\lambda\right)$ and the Eqn. (6) is multiplied by $P_n'\left(\lambda\right)$, the second result is substracted from the first, for $n\in\left\{0,1,...,N-1\right\}$ we obtain

$$a_{n-1}\left(P_{n-1}'\left(\lambda\right)P_n\left(\lambda\right)-P_n'\left(\lambda\right)P_{n-1}\left(\lambda\right)\right)-a_n\left(P_n'\left(\lambda\right)P_{n+1}\left(\lambda\right)-P_{n+1}'\left(\lambda\right)P_n\left(\lambda\right)\right)=\rho_nP_n^2\left(\lambda\right). \tag{9}$$

For $\lambda=\lambda_k$ and summing n from 0 to $N-1$, pay attention to Eqn. (7) and $P_N\left(\lambda_k\right)=0$ we have

$$a_{N-1}P_N'\left(\lambda_k\right)P_{N-1}\left(\lambda_k\right)=\sum_{j=0}^{M}P_j^2\left(\lambda\right)+\sum_{j=M+1}^{N-1}\alpha P_j^2\left(\lambda\right). \tag{10}$$

As a result, $P_N'\left(\lambda_k\right)\neq 0$.

According to Lemma 2, we can assume that $\lambda_1<\lambda_2<...<\lambda_N$. The following Lemma is about Parseval equality.

Lemma 3. The expansion formula which is equivalent Parceval equality, can be written as below:

$$\sum_{j=1}^{N}\frac{\eta}{\beta_j}P_m\left(\lambda_j\right)P_n\left(\lambda_j\right)=\delta_{mn},\quad m,n=\overline{0,N-1}, \tag{11}$$

where η is defined by

$$\eta = \begin{cases} 1, & m \text{ or } n \le M \\ \alpha, & m \text{ or } n > M \end{cases}, \tag{12}$$

and δ_{mn} is the Kronecker delta.

For $n = m = 0$ in the Eqn. (11) and from conditions (7) we obtain following equality

$$\sum_{j=1}^{N} \frac{1}{\beta_j} = 1. \tag{13}$$

Thus, we get eigenvalues $\{\lambda_k\}_{k=1}^{N}$ and eigenvectors v_j, $j = 1, 2, ..., N$ corresponding $\{\lambda_k\}_{k=1}^{N}$. So, we can say that the direct spectral problem of spectral analysis is solved.

Now let's try to answer the following question:

If we know eigenvalues $\{\lambda_k\}_{k=1}^{N}$ and eigenvectors $\{v_k\}_{k=1}^{N}$ of matrix J, is it possible to reconstruct the matrix J by using the following formula

$$Ju = \sum_{k=1}^{N} \lambda_k (u, v_k) v_k, \quad u \in l_2(0, N-1),$$

where $(u, v) = \sum_{j=0}^{N-1} u_j \bar{v}_j$ scalar product.

It is clear that eigenvalues of J is not sufficient for reconstruct matrix J. On account of this we must have some more information about eigenvectors.

Definition 4. The collection of quantitites $\{\lambda_k, \beta_k\}$ are called spectral data for the matrix J.

Additional we will need the presentation of entries of the matrix J by the polynomial $P_n(\lambda)$. For $\lambda = \lambda_j$ the Eqn. (6) is multiplied by $\frac{\eta}{\beta_j} P_m(\lambda_j)$, then summing by j from 1 to N and using Lemma 3, we have

$$a_n = \sum_{j=1}^{N} \frac{\eta^2 \lambda_j}{\beta_j} P_n(\lambda_j) P_{n+1}(\lambda_j), \quad n = \overline{0, \mathrm{N}-2} - \mathrm{M},$$

$$b_n = \sum_{j=1}^{N} \frac{\eta^2 \lambda_j}{\beta_j} P_n^2(\lambda_j), \quad n = \overline{0, \mathrm{N}-1},$$

where η is defined by (12). It is clear that $\rho_n = \eta = \alpha$ for m or $n > M$. Then, we can write these equalities as below:

$$a_n = \sum_{j=1}^{N} \frac{\lambda_j}{\beta_j} P_n(\lambda_j) P_{n+1}(\lambda_j), \quad n = \overline{0, M-1}, \tag{14}$$

$$a_M = \sum_{j=1}^{N} \frac{\alpha \lambda_j}{\beta_j} P_M(\lambda_j) P_{M+1}(\lambda_j), \quad c_M = \sum_{j=1}^{N} \frac{\lambda_j}{\beta_j} P_M(\lambda_j) P_{M+1}(\lambda_j), \tag{15}$$

$$c_n = \sum_{j=1}^{N} \frac{\alpha \lambda_j}{\beta_j} P_n(\lambda_j) P_{n+1}(\lambda_j), \quad n = \overline{M+1, N-2}, \tag{16}$$

$$b_n = \sum_{j=1}^{N} \frac{\lambda_j}{\beta_j} P_n^2(\lambda_j), \quad n = \overline{0, M}, \tag{17}$$

$$d_n = \sum_{j=1}^{N} \frac{\alpha \lambda_j}{\beta_j} P_n^2(\lambda_j), \quad n = \overline{M+1, N-1}. \tag{18}$$

3. Inverse Problem of Spectral Analysis

The inverse problem of spectral analysis is reconstruct matrix J by using the collection quantities $\{\lambda_k, \beta_k\}$.

Theorem 5. Let an arbitrary collection $\{\lambda_k, \beta_k\}$ of matrix is J. In order for this collection to be spectral data for some matrix which have form J, it is necessary and sufficient that the following conditions are satisfied:

(i) $\lambda_k \neq \lambda_j$, (ii) $\sum_{j=1}^{N} \frac{1}{\beta_j} = 1$, (iii) $a_n > 0, \ n = \overline{0, N-2}$.

Lemma 6. Let $\lambda_k,\ k=\overline{1,N}$ are distinct real numbers and for the positive numbers $\beta_k,\ k=\overline{1,N}$ be given that $\sum_{j=1}^{N}\frac{1}{\beta_j}=1.$ Then there exists unique polynomials $P_k(\lambda),\ k=\overline{0,N-1}$ with $\deg P_j(\lambda)=j$ and positive leading coefficients satisfying the conditions (11).

Now, we will give another method for a kind of approach to the solution of the inverse spectral problem which is called the Gelphand-Levitan-Marchenko method.

Let $R_n(\lambda)$ be a solution of the Eqn. (1) satisfying the conditions

$$R_{-1}(\lambda)=0,\quad R_0(\lambda)=1,$$

in the case $a_n\equiv 1,\ b_n\equiv 0$.

Recall that $P_n(\lambda)$ is a polynomial of degree n, so it can be expressed as

$$P_n(\lambda)=\gamma_n\left(R_n(\lambda)+\sum_{k=0}^{n-1}\chi_{n,k}R_k(\lambda)\right),\quad n\in\{0,1,...,M,...,N\}, \tag{19}$$

where γ_n and $\chi_{n,k}$ are constants. There is a connection between coefficients a_n,b_n,c_n,d_n and $\gamma_n,\chi_{n,k}$.

Then we can write the equalities

$$\begin{aligned} a_n&=\frac{\gamma_n}{\gamma_{n+1}}\quad (0\le n\le M),\quad \gamma_0=1,\\ c_n&=\frac{\gamma_n}{\gamma_{n+1}}\quad (M<n\le N-2),\ c_M=\frac{\gamma_M}{\alpha\gamma_{M+1}}, \end{aligned} \tag{20}$$

$$\begin{aligned} b_n&=\chi_{n,n-1}-\chi_{n+1,n}\quad (0\le n\le M),\qquad \chi_{0,-1}=0,\\ d_n&=\chi_{n,n-1}-\chi_{n+1,n}\quad (M<n\le N-1). \end{aligned} \tag{21}$$

Now, we can write from Eqn. (19)

$$\sum_{j=1}^{N}\frac{\eta}{\beta_j}P_n\left(\lambda_j\right)R_m\left(\lambda_j\right)=\gamma_n\left[G_{nm}+\sum_{k=0}^{n-1}\chi_{n,k}G_{km}\right], \tag{22}$$

where η is defined by (12) and

$$G_{nm}=\sum_{j=1}^{N}\frac{\eta}{\beta_j}R_n\left(\lambda_j\right)R_m\left(\lambda_j\right). \tag{23}$$

Since the expansion

$$R_j\left(\lambda\right)=\sum_{k=0}^{j}w_k^{(j)}P_k\left(\lambda\right)$$

holds, then from Eqn. (11) we have

$$\sum_{j=1}^{N}\frac{1}{\beta_j}P_n\left(\lambda_j\right)R_m\left(\lambda_j\right)=\frac{1}{\eta\gamma_n}\delta_{nm},\ n\geq 0,\ s=\overline{0,n}.$$

Considering the Eqn. (22) we get

$$G_{nm}+\sum_{k=0}^{n-1}\chi_{n,k}G_{km}=0,\quad m=\overline{0,n-1},\ n\geq 1, \tag{24}$$

$$G_{nn}+\sum_{k=0}^{n-1}\chi_{n,k}G_{kn}=\frac{1}{\eta\gamma_n^2},\quad n=\overline{0,\mathrm{N}-1}. \tag{25}$$

Eqn. (24) is important for the solution of inverse spectral problem. Firstly, G_{nm} are determined by using Eqn. (23) and then quantities $\chi_{n,k}$, $k=\overline{0,n-1}$ are found from system of Eqn. (24). Thus we can find unknowns γ_n with aid of $\chi_{n,k}$ from Eqn. (25).

Lemma 7. For any fixed n the system of Eqn. (24) is identically solvable.

Proof. It is clear that

$$v_n=\left(\frac{R_n\left(\lambda_1\right)}{\sqrt{\beta_1}},\frac{R_n\left(\lambda_2\right)}{\sqrt{\beta_2}},\ldots,\frac{R_n\left(\lambda_N\right)}{\sqrt{\beta_N}}\right),\ \ n=\overline{0,N-1},$$

are linear independent and from the Eqn. (23), we have

$G_{ns} = (v_n, v_s).$

The basic determinant of the system (24)

$$\det\left\{G_{ij}\right\}_{i,j=0}^{n-1} = \det\left\{\left(v_i, v_j\right)\right\}_{i,j=0}^{n-1}. \tag{26}$$

Lemma 8. Let G_{nm}, $m = \overline{0, n-1}$ be a solution of the system (24). Then

$$G_{nm} + \sum_{k=0}^{n-1} \chi_{n,k} G_{km} > 0, \quad n = \overline{0, \mathrm{N}-1}.$$

Proof. See [19].

Therefore we can determine the entries of the matrix J from the formulas Eqn. (20) and Eqn. (21).

4. Inverse Problem for Two Spectra

Consider the boundary value problem

$$a_{n-1} y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda \rho_n y_n, \quad n = \overline{1, N-1}, \tag{27}$$

with the boundary conditions

$$y_0 = y_N = 0, \tag{28}$$

where ρ_n is defined in (3). Now, the matrix of coefficients of Eqn. (27) is denoted by J_1 which has the same form with matrix J. If we delete the first row and the first column of the matrix J then we have $N-1 \times N-1$ tridiagonal matrix J_1 which has $N-1$ number eigenvalues μ_k, $k = \overline{1, N-1}$. Assume that eigenvalues of matrix J_1 are distinct and real. Thus we can write

$$\mu_1 < \mu_2 < \ldots < \mu_{N-1}.$$

The solution of Eqn. (27) is denoted by $\left\{Q_n(\lambda)\right\}$ provided that $Q_0(\lambda) = 0,\ Q_1(\lambda) = 1$. It is clear that the eigenvalues μ_j, $j = \overline{1, N-1}$ are zeros of the polynomial $Q_N(\lambda)$. While we determine entries of J, we will use eigenvalues of matrices J and J_1.

Now we will give an important lemma for the inverse spectral problem according to the two spectra.

Lemma 9. The eigenvalues of matrices J and J_1 alternate, i.e.

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < ... < \lambda_{N-1} < \mu_{N-1} < \lambda_N.$$

Proof. See [19].

Additionally, we can find the normalized numbers β_k by aid of two spectrums $\lambda_1, \lambda_2, ..., \lambda_N$ and $\mu_1, \mu_2, ..., \mu_{N-1}$ of the matrices J and J_1 respectively. Assume that

$$f_n(\lambda) = Q_n(\lambda) + m(\lambda) P_n(\lambda), \tag{29}$$

and require that $f_N(\lambda) = 0$. $m(\lambda)$ is a meromorphic function,

$$m(\lambda) = -\frac{Q_N(\lambda)}{P_N(\lambda)}, \tag{30}$$

its poles and zeros coincide with the eigenvalues of the problem (1)-(2) and (27)-(28), respectively. We see that the function $f_n(\lambda)$ satifies the equation

$$a_{n-1} f_{n-1}(\lambda) + b_n f_n(\lambda) + a_n f_{n+1}(\lambda) = \lambda \rho_n f_n(\lambda). \tag{31}$$

Now, if the equality Eqn. (31) is multiplied by $P_n(\lambda_k)$ and the Eqn. (6) (for $\lambda = \lambda_k$) is multiplied by $f_n(\lambda)$ then the second result is substracted from the first and sum by n, we obtain:

$$(\lambda - \lambda_k) \sum_{n=1}^{N-1} \rho_n f_n(\lambda) P_n(\lambda_k) = \sum_{n=1}^{N-1} \begin{Bmatrix} a_{n-1}\left(f_{n-1}(\lambda) P_n(\lambda_k) - P_{n-1}(\lambda_k) f_n(\lambda)\right) \\ -a_n\left(f_n(\lambda) P_{n+1}(\lambda_k) - P_n(\lambda_k) f_{n+1}(\lambda)\right) \end{Bmatrix}$$

or

$$(\lambda - \lambda_k) \sum_{n=1}^{N-1} \rho_n f_n(\lambda) P_n(\lambda_k) = -a_0,$$

and then for $\lambda \to \lambda_k$, we have

$$\beta_k = \frac{a_0 P_N'(\lambda_k)}{Q_N(\lambda_k)}.$$

On the other hand from Lemma 1, we get

$$\det(J-\lambda I) = (-1)^N a_0 a_1 ... a_M c_{M+1} ... c_{N-1} P_N(\lambda),$$

$$\det(J_1-\lambda I) = (-1)^{N-1} a_1 ... a_M c_{M+1} ... c_{N-1} Q_N(\lambda)$$

and from these equalities we can find

$$P_N(\lambda) = \frac{(\lambda-\lambda_1)...(\lambda-\lambda_N)}{a_0 a_1 ... a_M c_{M+1} ... c_{N-1}}, \qquad Q_N(\lambda) = \frac{(\lambda-\mu_1)...(\lambda-\mu_{N-1})}{a_1 ... a_M c_{M+1} ... c_{N-1}}.$$

As a result

$$\beta_k = \frac{\prod_{\substack{j=1 \\ j\neq k}}^{N}(\lambda_k-\lambda_j)}{\prod_{j=1}^{N-1}(\lambda_k-\mu_j)}. \tag{32}$$

Theorem 10. Let the collections $\{\lambda_k\}_{k=1}^N$, $\{\mu_k\}_{k=1}^{N-1}$ to be real numbers. These collections are spectrums of the $N\times N$ and $N-1\times N-1$ tridiagonal almost-symmetric matrices J and J_1, respectively, it is necessary and sufficient that they are alternate as below:

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < ... < \lambda_{N-1} < \mu_{N-1} < \lambda_N.$$

5. Conclusion

While solving the inverse spectral problem from two spectra, firstly we determine the normalized numbers β_k of the matrix J by using the eigenvalues $\{\lambda_k\}_{k=1}^N$, $\{\mu_k\}_{k=1}^{N-1}$ of the matrices J and J_1, respectively. Thus, we reduce the problem from two spectra to spectral analysis, and then we determine the values G_{nm}, $\chi_{n,\mathrm{k}}$ and γ_n from the formulas (23)-(25) by aid of the eigenvalues λ_k and the normalized numbers β_k of the matrix J.

Consequently, the entries a_n, b_n, c_n and d_n are found from the Eqn. (20) and Eqn. (21). Thus, the matrix J is reconstructed by using Parseval formula.

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